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19MEB302

HEAT & MASS TRANSFER

UNIT 1

CONDUCTION

CONTENTS

- One dimensional energy equation and boundary condition.
- Three dimension heat conduction equations
- Extended surface heat transfer
- Conduction with moving boundaries

CONDUCTION – STEADY STATE ONE DIMENSION

When the temperature of the body is a function only of radial distance and is independent of axial distance the systems like cylinder, sphere, may be treated as One-dimensional Systems.

In case of 2-D systems, the second space coordinate may be so small so it may be neglected and the multi-dimensional heat flow systems may be approximated into 1-D analysis and also the differential equations can be simplified, as a result of this simplification easy solutions are available.

FOURIER LAW OF HEAT CONDUCTION

Fourier law of heat conduction

Fourier's law states that the negative gradient of temperature and the time rate of heat transfer is proportional to the area at right angles of that gradient through which the heat flows. Fourier's law is the other name of the law of heat conduction.

Newton's law of cooling and Ohm's law are a discrete and electrical analog of Fourier's law.

$$q_x = -k A \frac{dT}{dx}$$

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

1. General heat conduction equation to Cartesian coordinates

Consider an infinitesimal rectangular volume element (parallelepiped) of sides dx, dy, dz parallel respectively to the three axes (X,Y,Z) in a medium in which temperature is varying with location & time.

Let ‘t’ = Temperature assumed uniform over the entire surface ‘ABCD’

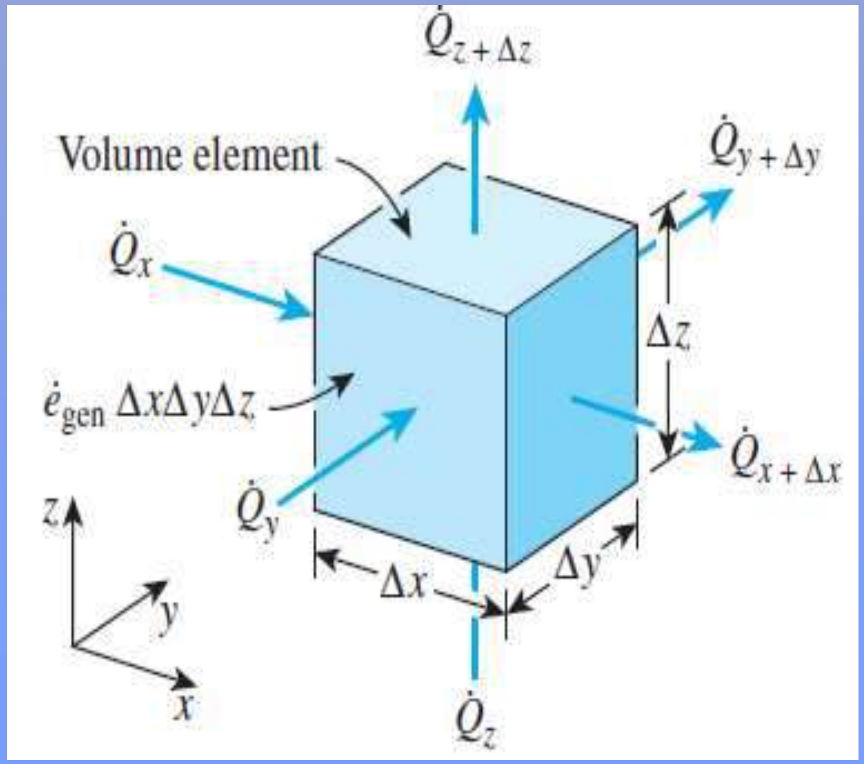
$\frac{dt}{dx}$ = Temperature changes and rate of change along x-direction

$\left(\frac{\partial t}{\partial x}\right) dx$ = Change of temperature through distance ‘dx’

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

$t + \left(\frac{\partial t}{\partial x}\right) dx$ = Temperature on the right face EFGH (at a distance ‘dx’ from the left face ABCD)

kx, ky, kz = thermal conductivities along X, Y, Z axes



GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

If the directional characteristics of a material are equal / same, it is called as Isotropic material and if $k_x \neq k_y \neq k_z$ Anisotropic material

Q_g = heat generation / unit volume/ unit time

Inside the control volume there may be heat sources due to flow of electric current in electric motors and generators, nuclear fission etc.

q_g may be function of position or time or both

ρ = mass density of material

c = specific heat of the material

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

Energy balance equation for volume element

Net heat accumulated in the element due to conduction of heat from all the directions (A)	+	heat generation within the element (B)	=	energy stored in the element (C)

A+B = C. here C = energy stored in the element

C = increase in internal energy / unit time + workdone by the element / unit time.

No work is done by the element / unit time

So A+B = Increase in internal energy/ unit time.

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

A = net heat accumulated the sum of

$$(Q_x - Q_x + d_x), (Q_y - Q_y + d_y) \text{ and } (Q_z - Q_z + d_z)$$

B = Internal heat generated (i.e. $q_g . d_x . d_y . d_z$)

C = Internal energy stored = $mC_p \Delta T$

Where:

M = mass of the element,

Cp = specific heat,

ΔT = change in temperature

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

Now

$$C = \rho.\, d_x.d_y.d_z C_p dT$$

$$\rho.C_p dT\,(d_x.d_y.d_z)$$

To find A:

Qx = the rate of heat flow into the element in ‘x’ direction through the face ABCD

$$Q_x = q_x.d_y.d_z = -k_x.\frac{\partial T}{\partial x}.\, d_y.d_z$$

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

Similarly in 'y' direction along face ABFE:

$$Q_y = q_y.d_x.d_z = -k_y.\frac{\partial T}{\partial y}.d_x.d_z$$

Similarly in 'z' direction along face DHEA:

$$Q_z = q_z.d_x.d_y = -k_z.\frac{\partial T}{\partial z}.d_x.d_y$$

Then the rate of heat flow in x direction through face 'x + dx', EFGH is

$$Q_{x+dx} = -k_x.\frac{\partial T}{\partial x}.d_y.d_z + \frac{\partial}{\partial x}\left(-k_x.\frac{\partial T}{\partial x}\right).d_x.d_y.d_z$$

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

Similarly in 'y' direction along face CDGH:

$$Q_{y+dy} = -k_y . \frac{\partial T}{\partial y} . d_x . d_z + \frac{\partial}{\partial y} \left(-k_y . \frac{\partial T}{\partial y}\right) . d_x . d_y . d_z$$

Similarly in 'z' direction along face CGBF:

$$Q_{z+dz} = -k_z . \frac{\partial T}{\partial z} . d_x . d_y + \frac{\partial}{\partial z} \left(-k_z . \frac{\partial T}{\partial z}\right) . d_x . d_y . d_z$$

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

Therefore the net flow of heat entering the element in x direction is the difference between entering and leaving heat flow rates, which is given by

$$Q_x - Q_{x+dx} = \frac{\partial}{\partial x}\left(-k_x.\frac{\partial T}{\partial x}\right).d_x.d_y.d_z \quad (a)$$

Similarly for:

$$Q_y - Q_{y+dy} = \frac{\partial}{\partial y}\left(-k_y.\frac{\partial T}{\partial y}\right).d_x.d_y.d_z \quad (b)$$

$$Q_z - Q_{z+dz} = \frac{\partial}{\partial z}\left(-k_z.\frac{\partial T}{\partial z}\right).d_x.d_y.d_z \quad (c)$$

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

The net heat conducted into the element is:

A = (a)+ (b)+ (c)

$$A = \left[\frac{\partial}{\partial x}\left(k_x.\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y.\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z.\frac{\partial T}{\partial z}\right) \right] d_x.d_y.d_z$$

$$B = q_g.\ d_x.d_y.d_z$$

$$C = \rho.C_p dT\,(d_x.d_y.d_z)$$

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

Since A + B = C

$$\left[\frac{\partial}{\partial x}\left(k_x.\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y.\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z.\frac{\partial T}{\partial z}\right) \right] + q_{gz} = \rho.C_p \frac{\partial T}{\partial x}$$

For most engineering problems

$$k_x = k_y = k_z$$

The general 3D equation becomes:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k} = \rho.C_p \frac{\partial T}{\partial t}$$

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

Where

$$\alpha = \frac{k}{\rho c}$$ = Thermal Diffusivity

α (larger) = faster the heat diffuse through the material & its temperature will change with time

α (larger) = for metals, solids, gases

α = important characteristics quantity for unsteady conduction situations

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

→ 3D Heat conduction equation

GENERAL HEAT CONDUCTION EQUATION TO CARTESIAN COORDINATES

For 1D → y=z=0, or vice versa

The equation may be written as

$$\nabla^2 t + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Other simplified forms of heat conduction equation in Cartesian coordinates

Other simplified forms of heat conduction equation in Cartesian coordinates :

1. When no heat source is present:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\nabla^2 t = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{(Fourier's Equation)}$$

Other simplified forms of heat conduction equation in Cartesian coordinates

2. When temperature does not depend on time, the conduction then takes place in steady state.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k} = 0$$

$$\nabla^2 t + \frac{q_g}{k} = 0$$ (Poisson's Equation)

In the absence of heat generation

$$\nabla^2 t = 0$$ (Laplace's Equation)

Other simplified forms of heat conduction equation in Cartesian coordinates

3. Steady state 1-D heat transfer:

$$\frac{\partial^2 T}{\partial x^2} + \frac{q_g}{k} = 0$$

4. Steady state 1-D without internal heat generation:

$$\frac{\partial^2 T}{\partial x^2} = 0$$

5. Steady state 2-D without internal heat generation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Other simplified forms of heat conduction equation in Cartesian coordinates

6. Unsteady state 1-D without internal heat generation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

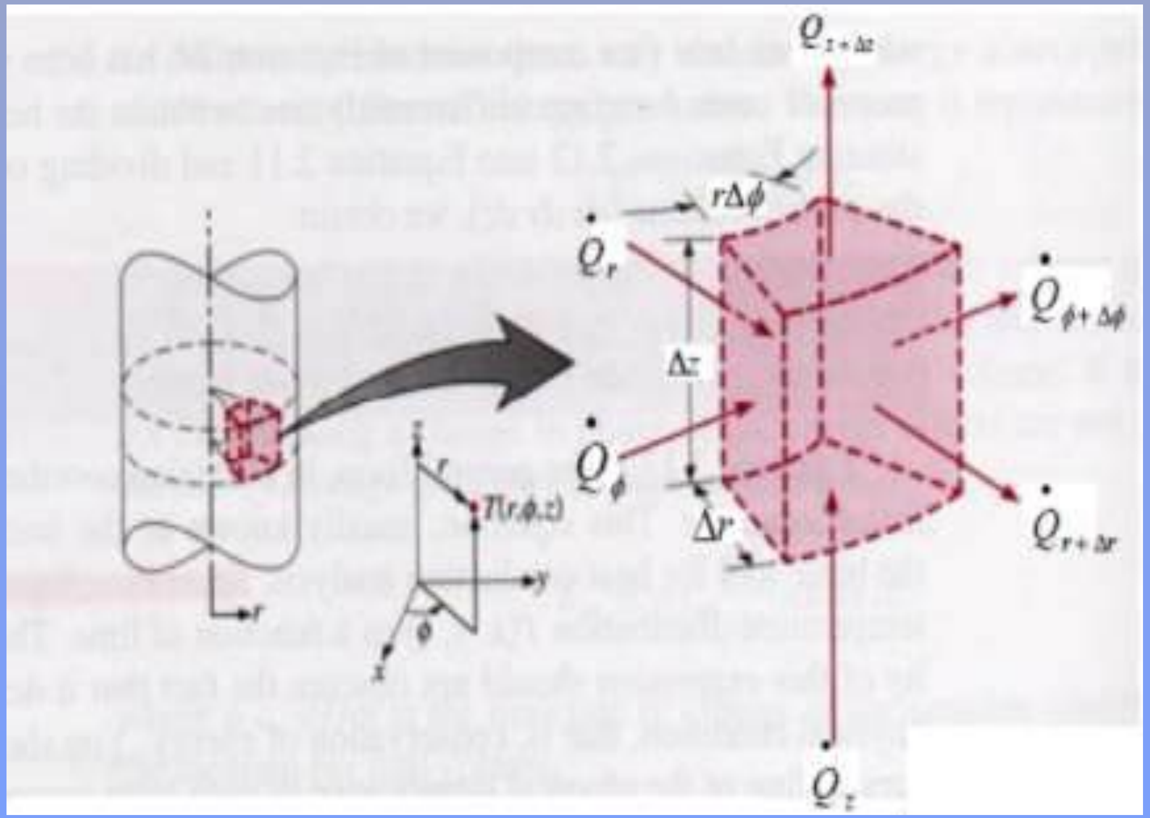
Heat conduction equation in cylindrical coordinates:

Consider a small volume of sides dr, rdΦ, dz

Assume the material to be isotropic.

The rate of heat flow into the element in r direction:

$$Q_r = \frac{-k \cdot \partial T}{\partial r} \cdot r d\Phi \cdot dz$$



In (r + dr) direction

$$Q_{r+dr} = -Q_r + \frac{\partial Q_r}{\partial r} \cdot d_r$$

HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

The net rate of heat entering the element in 'r'-direction is:

$$Q_r + Q_{r+dr} = Q_r - Q_r. + \frac{\partial Q_r . d_r .}{\partial r}$$

$$- \frac{\partial Q_r . d_r .}{\partial r}$$

$$\frac{-\partial}{\partial r}\left(-k . \frac{\partial T}{\partial r} . r d\Phi, dz \right) d_r .$$

$$k . \frac{\partial}{\partial r}\left(r \frac{\partial T .}{\partial r} \right) d_r . d\Phi, dz$$

HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

$$Q_r - Q_{r+dr} = k.\left(\frac{\partial^2 T}{\partial r} + r \frac{\partial T}{\partial r} \right) d_r . d\Phi, dz \quad ...(1)$$

Then for:

$$Q_\Phi = - k.\frac{\partial T}{r\,\partial \Phi} . d_r . d_z$$

$$Q_{\Phi + d\Phi} = Q_\Phi + \frac{\partial Q_\Phi . r d_\Phi .}{r\,\partial \Phi}$$

HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

$$Q_\Phi - Q_{\Phi + d\Phi} = - \frac{\partial Q_\Phi . r d\Phi .}{r \, \partial \Phi}$$

$$Q_\Phi - Q_{\Phi + d\Phi} = - \frac{\partial}{r \, \partial \Phi} \left(- k . \frac{\partial t}{r \, \partial \Phi} dr . dz\right) r d\Phi$$

$$\frac{k}{r} \frac{\partial}{\partial \Phi} \left(\frac{\partial t}{r \, \partial \Phi} \right) dr . dz \; r d\Phi$$

HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

$$\frac{k}{r^2}\left(\frac{\partial^2 t}{\partial \Phi^2}\right) dr.\,dz\; rd\Phi$$

$$k\left(\frac{1}{r^2}\;\frac{\partial^2 t}{\partial \Phi^2}\right) dr.\,dz\; rd\Phi \quad \ldots(2)$$

HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

In z- direction

$Q_z - Q_{z+dz}$

$$Q_z = \left(-k \frac{\partial T}{\partial z}\right) . dr. dz\ rd\Phi$$

$$Q_{z+dz} = - Q_z. + \frac{\partial Q_z}{\partial z} . d_z.$$

$$Q_z - Q_{z+dz} = Q_z - \left(Q_z - \frac{\partial Q_z . d_z.}{\partial z}\right)$$

HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

$$k\left(\frac{\partial^2 t}{\partial \Phi^2}\right) rd_\Phi \, dr.\, dz \quad ..(3)$$

The net heat conducted into the element rd_Φ dr. dz / unit time

$$k\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} + \frac{\partial t}{\partial r}\right) + k\left[\frac{1}{r^2}\frac{\partial^2 t}{\partial^2 \Phi}\right] + k.\frac{\partial^2 t}{\partial z^2} \quad rd_\Phi \, dr.\, dz \quad ....(A)$$

Since A + B = C

Net heat conducted + internal heat generated = change in internal energy / unit time / unit time / unit time

HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

$$B = q_g = r d_\Phi\, dr.\, dz$$

$$C = \rho\, C_p \frac{\partial t}{\partial r} r d_\Phi\, dr.\, dz$$

From A, B, C take out rd_Φ dr. dz and divide by K.

Then A+B = C becomes

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial^2 \Phi} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial z}$$

HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES SYSTEM

Heat conduction equation in Spherical Coordinates System:

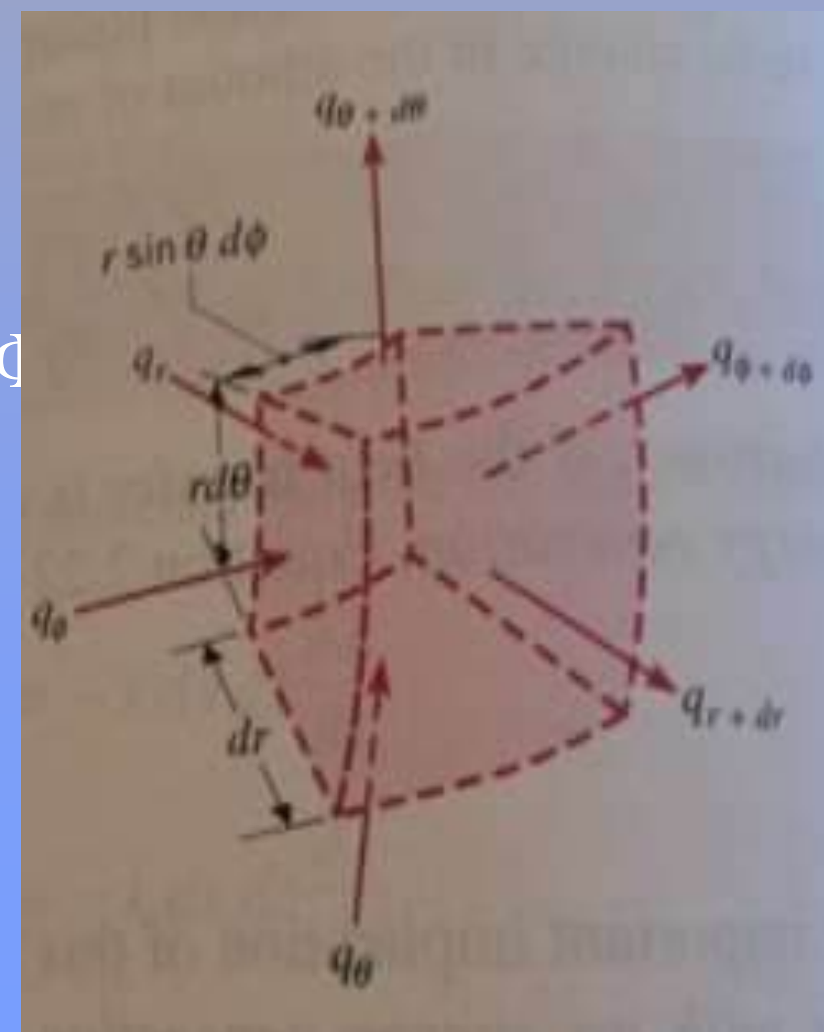
Consider an infinitesimal spherical element of an isotropic material having the coordinates (r, Φ, Ψ). The sides of the element are dr, rd Ψ & r sinΨ dΦ.

Energy balance equation (A+ B = C)

$A = (Q_r - Q_r + dr) + (Q_\Psi - Q_\Psi + d\Psi) + (Q_\Phi - Q_\Phi + d\Phi$

$B = q_g . dr . rd\Psi . r\sin\Psi . d\Phi$,

$$C = \rho C_p \frac{\partial t}{\partial r} dr.rd\Psi.r\sin\Psi.d\Phi$$

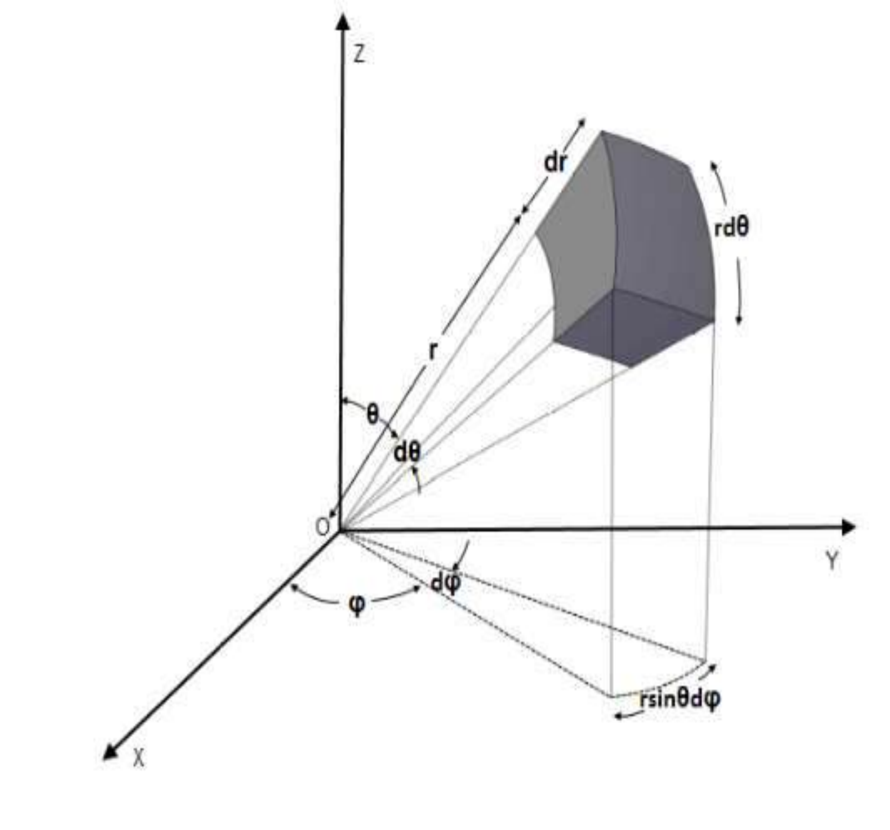


HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES SYSTEM

Now, the rate of heat flow in y direction

$$Q_r = -k \frac{\partial t}{\partial r} .rd\Psi.r\sin\Psi.d\Phi$$

$$Q_{r+dr} = Q_r + \frac{\partial Q_r}{\partial r} dr$$



The rate of heat flow out of the element in r- direction

$$Q_r - Q_{r+dr} = Q_r - \left(Q_r + \frac{\partial Q_r}{\partial r} dr\right)$$

HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES SYSTEM

$$- \frac{\partial}{\partial r^2} \left(-k \frac{\partial t}{\partial r} r d\Psi . r \sin\Psi . d\Phi\right) . dr$$

$$k \frac{\partial}{\partial r^2} \left(r^2 \frac{\partial t}{\partial r} \right) r d\Psi . r \sin\Psi . d\Phi \; . dr$$

Multiply and divide by r^2

$$k \left(\frac{\partial^2 t}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right) r d\Psi . r \sin\Psi . d\Phi \; . dr \qquad \text{.......(1)}$$

HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES SYSTEM

Similarly in z- direction (Ψ direction)

$$Q_{\Psi} = -k \frac{\partial t}{r \partial \Psi} r d\Psi . r \sin\Psi . d\Phi$$

Rate of heat flow out of $rd\Psi$, the element in Ψ direction

$$Q_{\Psi} + Q_{\Psi + d\Psi} = - \frac{\partial Q_{\Psi}}{r \partial_{\Psi}} r d\Psi .$$

HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES SYSTEM

$$= - \frac{\partial Q_\Psi}{r\partial_\Psi} rd\Psi.$$

$$- \frac{\partial}{r\partial_\Psi}\left(- k \frac{\partial t}{r\partial_\Psi} . dr .r\sin\Psi.d\Phi \right) rd\Psi$$

$$\frac{k}{r^2} \frac{\partial}{\partial_\Psi}\left(\sin\Psi \frac{\partial T}{\partial_\Psi} \right) dr.rd\Phi.rd\Psi$$

HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES SYSTEM

Multiply & divide by sin Ψ

$$\frac{k}{r^2 \sin\Psi} \frac{\partial}{\partial_\Psi}\left(\sin\Psi \frac{\partial T}{\partial_\Psi} \right) dr.rd\Psi.r\sin\Psi.d\Phi \quad \text{.....(2)}$$

In Φ direction (x- direction)

$$Q_\Phi = -k \frac{\partial T}{r\sin\Psi\,\partial\Phi} dr.rd\Psi.$$

HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES SYSTEM

$$Q_\Phi - Q_{\Phi+d\Phi} = -\frac{\partial Q_\Phi}{r\sin\Psi\,\partial\Phi}\, r\sin\Psi . d\Phi$$

$$-\frac{\partial}{r\sin\Psi\,\partial\Phi}\left[-k\frac{\partial T}{r\sin\Psi\,\partial\Phi}\, dr.rd\Psi.\right] r\sin\Psi . d\Phi$$

$$k\frac{1}{r^2\sin^2\Psi}\frac{\partial^2 T}{\partial\Phi^2}\, dr.rd\Psi.\, r\sin\Psi . d\Phi \quad \ldots..(3)$$

Add (1) +(2) + (3)

HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES SYSTEM

A =

$$k\left[\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right] dr.rd\Psi.\ r\sin\Psi.d\Phi + \frac{k}{r^2\sin\Psi}\frac{\partial}{\partial\Psi}\left(\sin\Psi\ \frac{\partial T}{\partial\Psi}\right) dr.rd\Psi.r\sin\Psi.d\Phi\ +$$

$$k\ \frac{1}{r^2\sin^2\Psi}\frac{\partial^2 T}{\partial\Phi^2}\ dr.rd\Psi.\ r\sin\Psi.d\Phi$$

B = $q_g\ dr.rd\Psi.r\sin\Psi.d\Phi$

C = $\rho\, C_p \frac{\partial T}{\partial t} dr.rd\Psi.r\sin\Psi.d\Phi$

The net heat conducted into the element dr.rdΨ.rsinΨ.dΦ / unit time is

HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES SYSTEM

The net heat conducted into the element dr.rdΨ.rsinΨ.dΦ / unit time is

$$k\left[\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2\sin^2\Psi}\frac{\partial}{\partial \Psi}\left(\sin\Psi\frac{\partial T}{\partial \Psi}\right)+\frac{1}{r^2\sin^2\Psi}\frac{\partial^2 T}{\partial \Phi^2}\right]+q_g=\rho C_p\frac{\partial T}{\partial t}$$

Written as:

$$\left[\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2\sin^2\Psi}\frac{\partial}{\partial \Psi}\left(\sin\Psi\frac{\partial T}{\partial \Psi}\right)+\frac{1}{r^2\sin^2\Psi}\frac{\partial^2 T}{\partial \Phi^2}\right]+\frac{q_g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES SYSTEM

For steady state 1 D heat conduction in radial direction without heat generation.

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = 0$$

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial T}{\partial r} \right] = 0$$

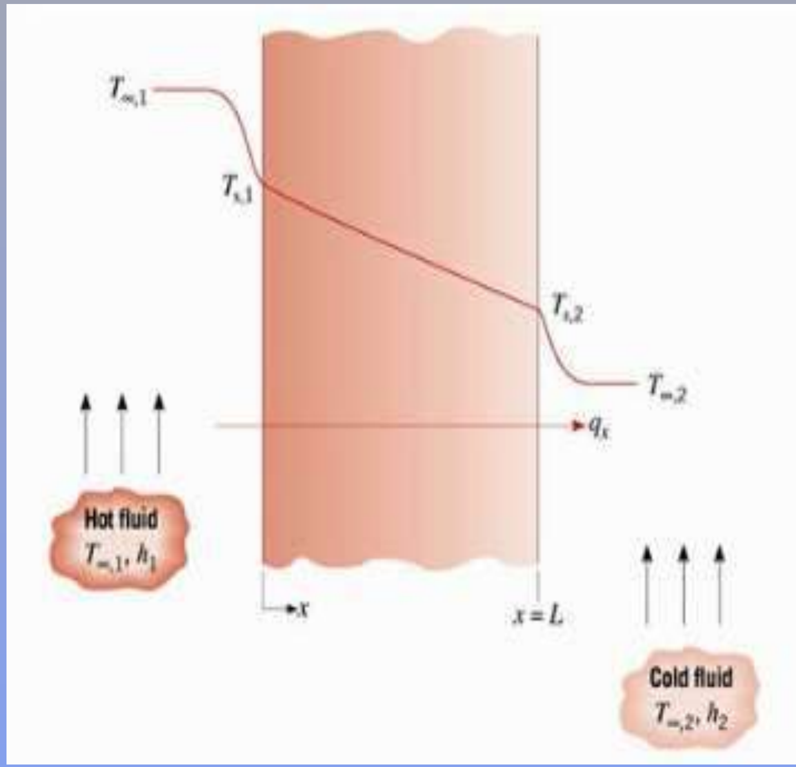
HEAT CONDUCTION THROUGH PLANE WALL

Case (1): Uniform Thermal Conductivity

Case (2): Variable Thermal Conductivity

Case (1): Uniform Thermal Conductivity

Uniform Thermal Conductivity:



Assumptions:

Wall – plane wall

Material – homogenous

Heat flow – x- direction

Let:

L – Thickness of the plane wall.

A – cross section area of the wall.

K – thermal conductivity of the wall material

T1, T2 – Temperatures maintained at the faces 1 & 2.

General heat conduction equation in Cartesian Coordinates:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t} = 0$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial z^2} = 0$$

$$q_g = 0$$

$$\frac{\partial^2 T}{\partial x^2} = 0$$

To find the solution for equation a, it requires two BC's since it is a 2nd order DE's.

BC's are

$T = T_1$ at $x = 0$;

$T = T_2$ at $x = L$;

Integrating equation (A) twice

$$\frac{dt}{dx} = c_1 \qquad T = C_1 x + C_2 \quad \text{...(B)}$$

To find C_1 & C_2 from BC's

$x = 0, T = T_1;$ $C_2 = T_1$

$x = L, T = T_2;$

$T_2 = C_1 L + C_2$

$C_1 L + T_1$

$T_2 - T_1 = C_1 L$

$$\frac{T_2 - T_1}{L} = C_1$$

Sub C_1 & C_2 in (B)

$$T = \left(\frac{T_2 - T_1}{L} \right) x + T_1$$

Differentiate the above equation

$$\frac{dT}{dx} = \left(\frac{T_2 - T_1}{L} \right)$$

we know

$$- k. A. \frac{dt}{dx}$$

$$- k. A. \left(\frac{T_2 - T_1}{L} \right)$$

Thermal resistance of the wall =

$$R_{th} = \frac{L}{kA} \quad ....(1)$$

Weight of the wall = ρ.A.L ....(2)

Sub the value of L in (2)

$$\mathbf{W = (\rho.k).A^2.R_{th}}$$

The lightest insulation will be one which has small product of density (ρ) & (k)

Case (2): Variable Thermal Conductivity

1. Temperature variation in terms of surface temperature (t1, t2)
2. Temperature variation in terms of heat flux (q)

1. Temperature variation in terms of surface temperature (t1, t2):

let the thermal conductivity vary with temperature according to the relation.

$$K = k_o (1 + \beta t)$$

When the effect of thermal conductivity is considered.

$$Q = -k.A.\frac{dt}{dx}$$

$$Q = -k_o (1 + \beta t) \frac{dt}{dx} A$$

$$\frac{Q}{A}\,dx = -k_o(1+\beta t)\,dt$$

$$\int_0^L dx = -k_o\int_{t1}^{t2}(1+\beta t)\,dt$$

$$\frac{Q}{A}(L-0) = -k_o\left[t+\frac{\beta t^2}{2}\right]_{t1}^{t2}$$

$$k_o \left[(t_2 - t_1) + \frac{\beta}{2} (t_2^2 - t_1^2) \right]$$

$$Q = k_o (t_2 - t_1) \left[1 + \beta \frac{(t_1 + t_2)}{2} \right]$$

$$Q = k_o [1 + \beta t_m] . A \frac{(t_1 - t_2)}{L}$$

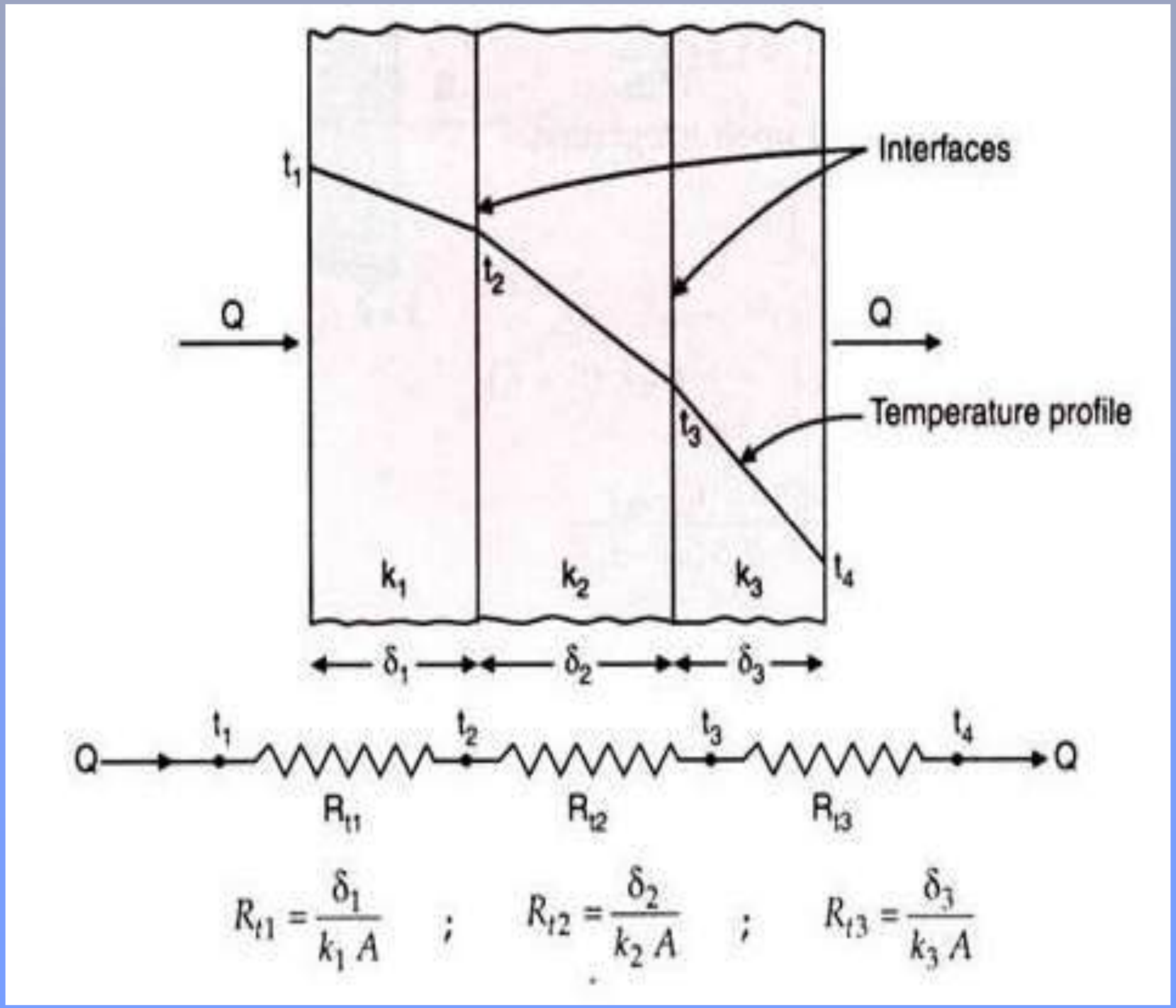
$$Q = k_m . A \frac{[t_1 - t_2]}{L}$$

k_m = mean thermal conductivity of the wall material

HEAT CONDUCTION THROUGH A COMPOSITE WALL

Heat conduction through a composite wall:

consider the transmission of heat through a composite wall consisting of a number of slabs.



HEAT CONDUCTION THROUGH A COMPOSITE WALL

Assuming that there is a perfect contact between the layers and no temperature drop occurs the interface between the materials.

$$Q = \frac{k_1 A}{\delta_1}(t_1 - t_2) = \frac{k_2 A}{\delta_2}(t_2 - t_3) = \frac{k_3 A}{\delta_3}(t_3 - t_4)$$

Rewriting the above expression in terms of temperature drop across each layer,

$$t_1 - t_2 = \frac{Q\delta_1}{k_1 A}\ ;\ t_2 - t_3 = \frac{Q\delta_2}{k_2 A}\ ;\ t_3 - t_4 = \frac{Q\delta_3}{k_3 A}$$

Summation gives the overall temperature difference across the wall

$$t_1 - t_4 = Q\left(\frac{\delta_1}{k_1 A} + \frac{\delta_2}{k_2 A} + \frac{\delta_3}{k_3 A}\right)$$

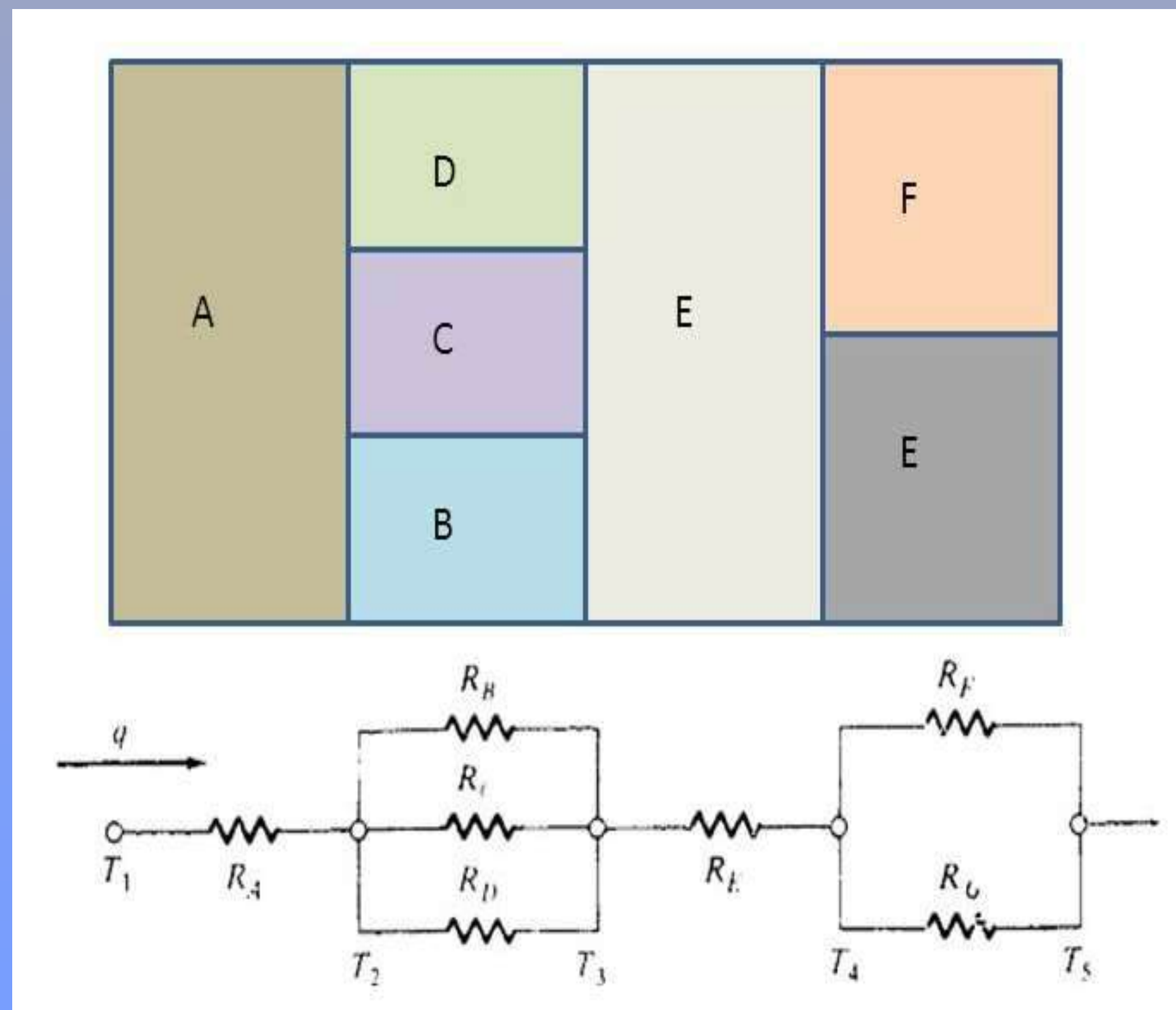
Then

$$Q = \frac{(t_1 - t_4)}{\delta_1/k_1 A + \delta_2/k_2 A + \delta_3/k_3 A} = \frac{(t_1 - t_4)}{R_{t_1} + R_{t_2} + R_{t_3}}$$

$$Q = \frac{\Delta t}{\sum R_t}$$

If the wall consists of both Parallel and Series Resistances, the electrical analogy may be used.

$$Q = \frac{\Delta t}{\sum R_t}$$



OVERALL HEAT RESISTANCE

It is the heat transmitted per unit area per unit time per degree temperature difference between the bulk fluids on each sides of the metal.

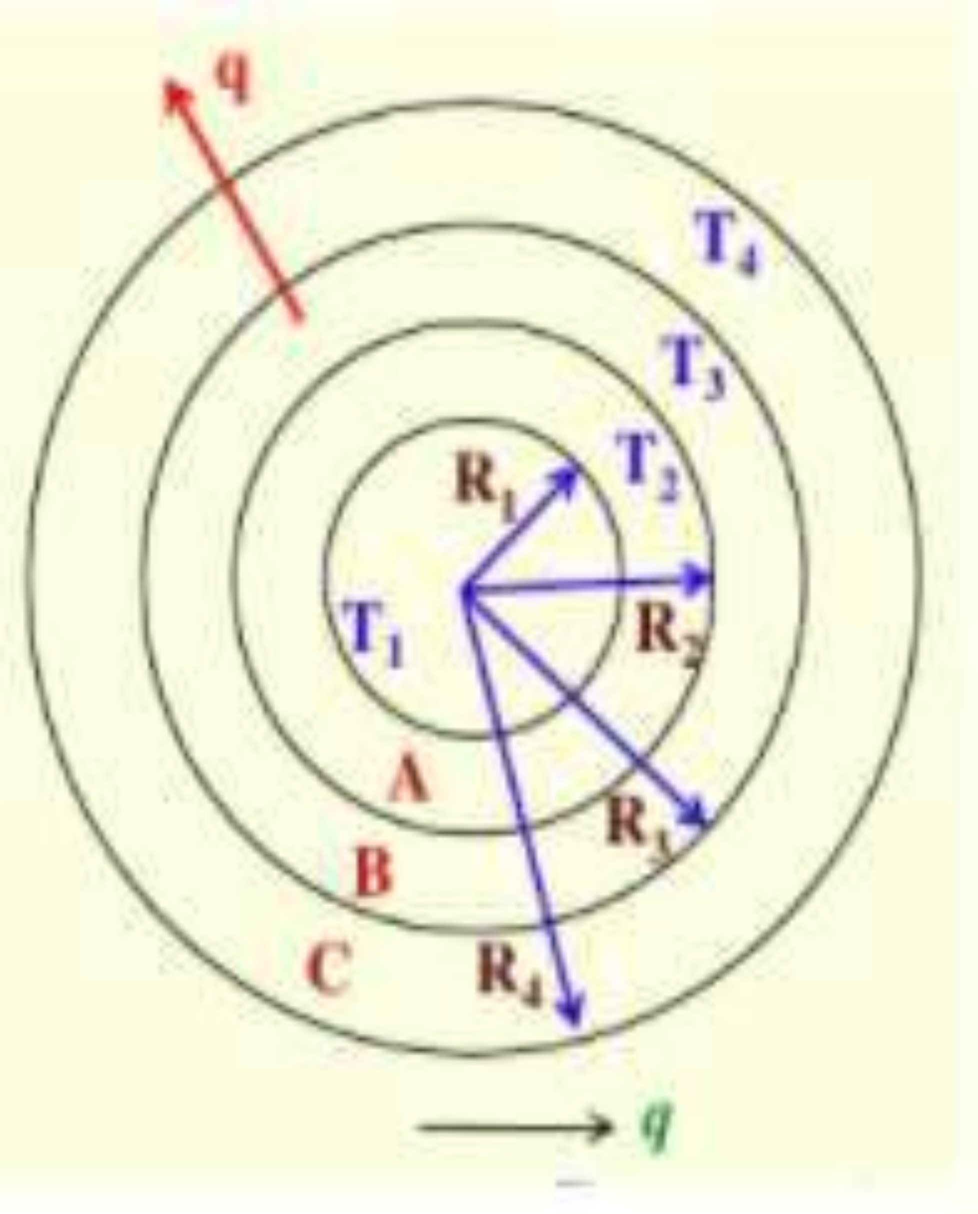
$$Q = U.A.\Delta T \quad (W/m^2K)$$

HEAT TRANSFER THROUGH COMPOSITE PIPES WITH INSIDE AND OUTSIDE CONVECTION

Temperature of hot fluid T_a,
Heat transfer coefficient flowing through pipe h_a,
Separated by two layers from atmosphere.

Thermal conductivity of 1^{st} layer – k_1,
Thermal conductivity of 2^{nd} layer – k_2,

Outside surface heat is being transferred to a cold
fluid at temperature T_b, heat transfer coefficient h



Heat transfer by convection at side ‘A’

$$Q = h_a . A . [T_a - T_1]$$

$$Q = 2\Pi r_1 L\, h_a [T_a - T_1]$$

Heat transfer by conduction at section '1' is

$$Q = \frac{2\Pi k_1 L [T_a - T_1]}{\ln\left(\frac{r2}{r1}\right)}$$

Similarly at section ‘2’

$$Q = \frac{2\Pi k_1 L [T_2 - T_3]}{\ln\left(\frac{r_3}{r_2}\right)}$$

Heat transfer by convection at side ‘B’ is

$$Q = h_b . A. [T_a - T_b]$$

$$Q = 2\Pi r_3 L\, h_b [T_3 - T_b]$$

$$T_a - T_1 = \frac{Q}{2\Pi r_1 h_a}$$

$$T_1 - T_2 = \frac{Q}{2.\Pi.L.k_1} \times \ln\left(\frac{r_2}{r_1}\right)$$

$$T_2 - T_3 = \frac{Q}{2.\Pi.L.k_2} \times \ln\left(\frac{r_3}{r_2}\right)$$

$$T_3 - T_b = \frac{Q}{2.\Pi.r_3.\,h_b}$$

Add all above equations on both sides:

$$T_a - T_b = \frac{Q}{2.\Pi.L} \quad \frac{1}{h_a r_1} \quad \frac{\ln\left(\frac{r_2}{r_1}\right)}{\substack{k_1 \\ r_3}} \quad \frac{\ln\left(\frac{r_3}{r_2}\right)}{k_2} \quad \frac{1}{h_b r_3}$$

$$Q = \frac{T_a - T_b}{\frac{1}{2.\Pi.L} \quad \frac{1}{h_a r_1} \quad \frac{\ln\left(\frac{r_2}{r_1}\right)}{k_1} \quad \frac{\ln\left(\frac{r_3}{r_2}\right)}{k_2} \quad \frac{1}{h_b r_3}}$$

$$Q = \frac{\Delta T}{R}$$

$$R = \frac{1}{UA}$$

$$Q = UA\,[T_a - T_b]$$

Problem:

Calculate the rate of heat loss from a red brick wall of length 5m, height 4m, thickness 0.25m. The temperature of the inner surface is 110°C and that of the outer surface is 40°C. The thermal conductivity of red brick, k= 0.70 W/mK. Calculate also the temperature at an interior point of the wall 20 cm distance from the inner wall.

Solution:

$$Q = \frac{k.A.(T_1 - T_2)}{L}$$

$$Q = \frac{0.7 \times (5 \times 4) \times (110 - 40)}{0.25}$$

$$Q = 3920\ W \quad (or) \quad 3.92\ kW$$

At x = 0.2

$$\left[\frac{T_2 - T_1}{L} \right] * x + T_1$$

$$\left[\frac{40 - 110}{0.2} \right] * 0.2 + 110$$

$$\left[\frac{-70}{0.25} \right] * 0.2 + 110$$

$$T_1 \quad = -56 + 110 = 54^{o}C$$

Problem:

A wall of furnace is made up of inside layer of silica brick 120 mm thick covered with a layer of magnesite brick 240 mm thick. The temperature at the inside surface of silica brick wall and outside surface of magnesite brick wall are 752^{o}C and 110^{o}C respectively. The contact thermal resistance between the two walls at the interface is 0.0035/W per unit wall area. If thermal conductivities of silica and magnesite bricks are 1.7 W/m^{o}C and 5.8 W/m^{o}C, Calculate:

1. Rate of heat loss per unit area of walls
2. Temperature drop at the interface

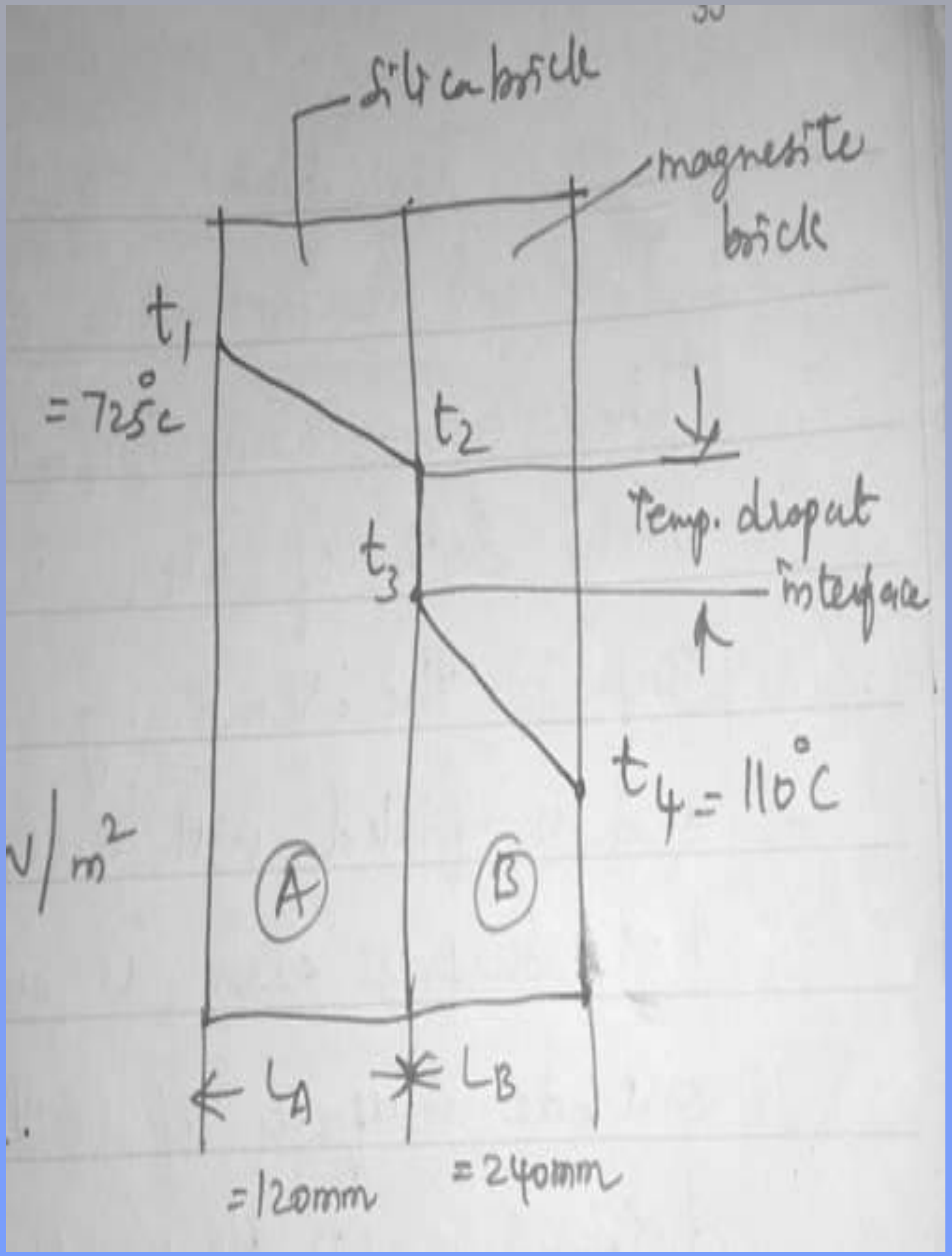
Solution:

(1)

$$q = \frac{\Delta T}{\Sigma R_{th}}$$

$$= \frac{t_1 - t_4}{R_{th\,A} + R_{th\,Const} + R_{th\,B}}$$

$$= \frac{725 - 110}{\frac{L_A}{K_A} + 0.0035 + \frac{L_B}{K_B}}$$



$$= \frac{615}{\frac{0.12}{1.7} + 0.0035 + \frac{0.24}{5.8}}$$

$$= \boxed{q = 5324.67\ \text{W/m}^2}$$

(2)

$$t_2 - t_3 \quad q = \frac{t_1 - t_2}{\frac{L_A}{K_A}} = \frac{t_3 - t_4}{\frac{L_B}{K_B}}$$

To find t_2

$$q = \frac{t_1 - t_2}{\frac{L_A}{K_A}} = 5324.67 \text{ W/m}^2 \; ; \quad \boxed{t_2 = 349.14\,^{o}\text{C}}$$

To find t_3:

$$\frac{t_3 - 110}{\frac{0.24}{5.8}} = \quad t_3 = 330.33\ ^{o}C$$

$$t_2 - t_3 = 18.81\ ^{o}C$$

Answers:

(1)

$$q = 5324.67\ W/m^2$$

(2)

$$t_2 - t_3 = 18.81\ ^{o}C$$

Problem:

Two slabs each 120 mm thick, have thermal conductivities of 14.5 W/m°C and 210 W/m°C. These are placed in contact, but due to roughness, onl 30% of area is in contact and the gap in the remaining area is 0.025 mm thick and is filled with air. If the temperature of the hot surface is at 220°C and the outside surface of other slab is at 30°C, determine:

1. Heat flow through the composite system
2. Contact resistance and temperature drop in contact

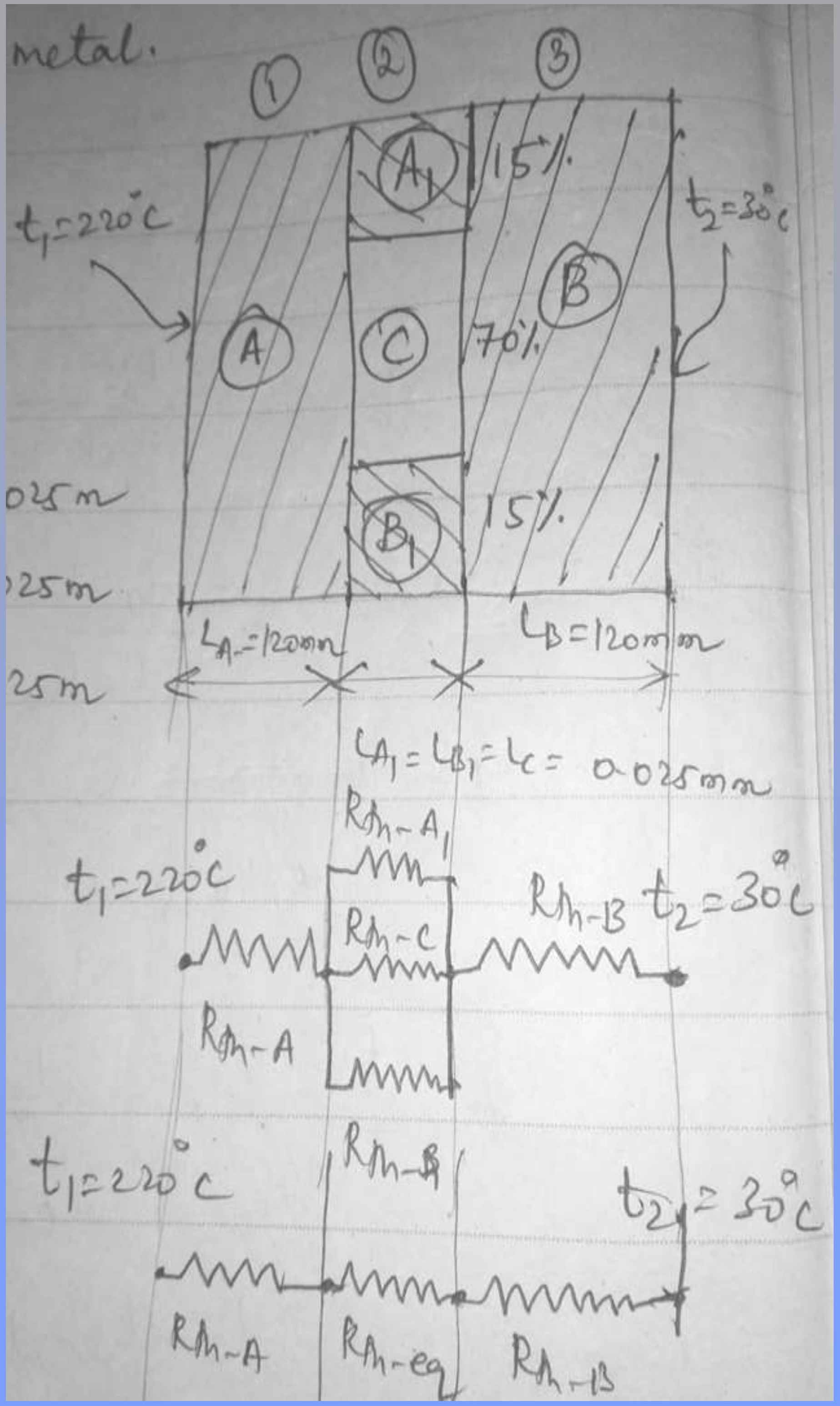
Assume that the conductivity of air is 0.032 W/m°C and that half of the contact (of the contact area) is due to either metal.

Solution:

(1) $$Q = \frac{(\Delta T)_{overall}}{(R_{th})_{total}}$$

$$= \frac{220 - 30}{(R_{th})_{total}}$$

$$(R_{th})_{total} = R_{th\,A} + R_{th\,Eq} + R_{th\,B}$$ (D.B Pg: 47)



$$\frac{L_A}{K_A A_A} + \frac{R_{A1}.\ R_{B1}.\ R_{C1}.}{R_{A1}.\ R_{B1}. + R_{B1}.\ R_{C1} + R_{A1}.\ R_{C1}.} + \frac{L_B}{K_B\ A_B}$$

$$R_1 = \frac{L_A}{K_A A_A}$$

$$= \frac{0.12}{14.5 \text{ x } 1} =$$

$$R_1 = 0.0082 = 8.2 \text{ x } 10^{-3}$$

R_2

$$(R_{th})_{A1} = \frac{L_{A1}}{K_{A1} . A_{A1}} \rightarrow \frac{0.000025}{14.5 \text{ X } 0.15} = 1.1 \text{ X } 10^{-7}$$

$$(R_{th})_{B1} = \frac{L_{B1}}{K_{B1} . A_{B1}} \rightarrow \frac{0.000025}{210 \text{ X } 0.15} = 7.93 \text{ X } 10^{-7}$$

$$R_{C} = \frac{L_C}{K_C . A_C} \rightarrow \frac{0.000025}{0.032 \text{ X } 0.7} = 1.11 \text{ X } 10^{-3}$$

$$R_2 = \frac{1.11 \text{ X } 10^{-3} \text{ x } 1.11 \text{ X } 10^{-7} \text{ x } 7.93 \text{ X } 10^{-7}}{1.1 \text{ x } 7.93 \text{ x } 10^{-14} \text{ x } 7.93 \text{ x } 1.1 \text{ x } 10^{-10} \text{ x } 1.1 \text{ x } 1.1 \text{ x } 10^{-10}}$$

$$R_2 = 9.7 \text{ x } 10^{-8}$$

R_3 → $\dfrac{0.12}{210 \times 1} = 5.7 \times 10^{-3}$

$$(R_{th})_{total} = R_{1.} + R_{2.} + R_{3.}$$

$$8.2 \times 10^{-3} + 9.7 \times 10^{-8} + 5.7 \times 10^{-4}$$

$$(R_{th})_{total} = 8.77 \times 10^{-3}$$

$$Q = \frac{(\Delta T)_{overall}}{(R_{th})_{total}}$$

$$\frac{190}{8.77 \times 10^{-3}} = 21\ kW$$

(ii) Contact Resistance = 9.7×10^{-8} ºC/ W

Temperature drop in contact = Q x Contact Resistance

$= 21000 \times 9.7 \times 10^{-8}$

$= 0.2 \times 10^{-2}$ (or) 0.015 ºC

Problem:

A cold storage rooms has walls made of 220 mm of brick on the outside, 90mm of plastic foam and finally of 16mm of wood on the inside. The outside and inside temperature are 25 ºC & -3 ºC. If

$h_i = 30\ W/m^2\ {}^oC$

$h_{out} = 11\ W/m^2\ {}^oC$

$k_{brick} = 0.99\ W/m^2\ {}^oC$

$k_{foam} = 0.022\ W/m^2\ {}^oC$

$k_{wood} = 0.17\ W/m^2\ {}^oC$

Determine :

1. Rate of heat removal by refrigeration if the total wall area is 85 m^2
2. Temperature of the inside surface of the brick

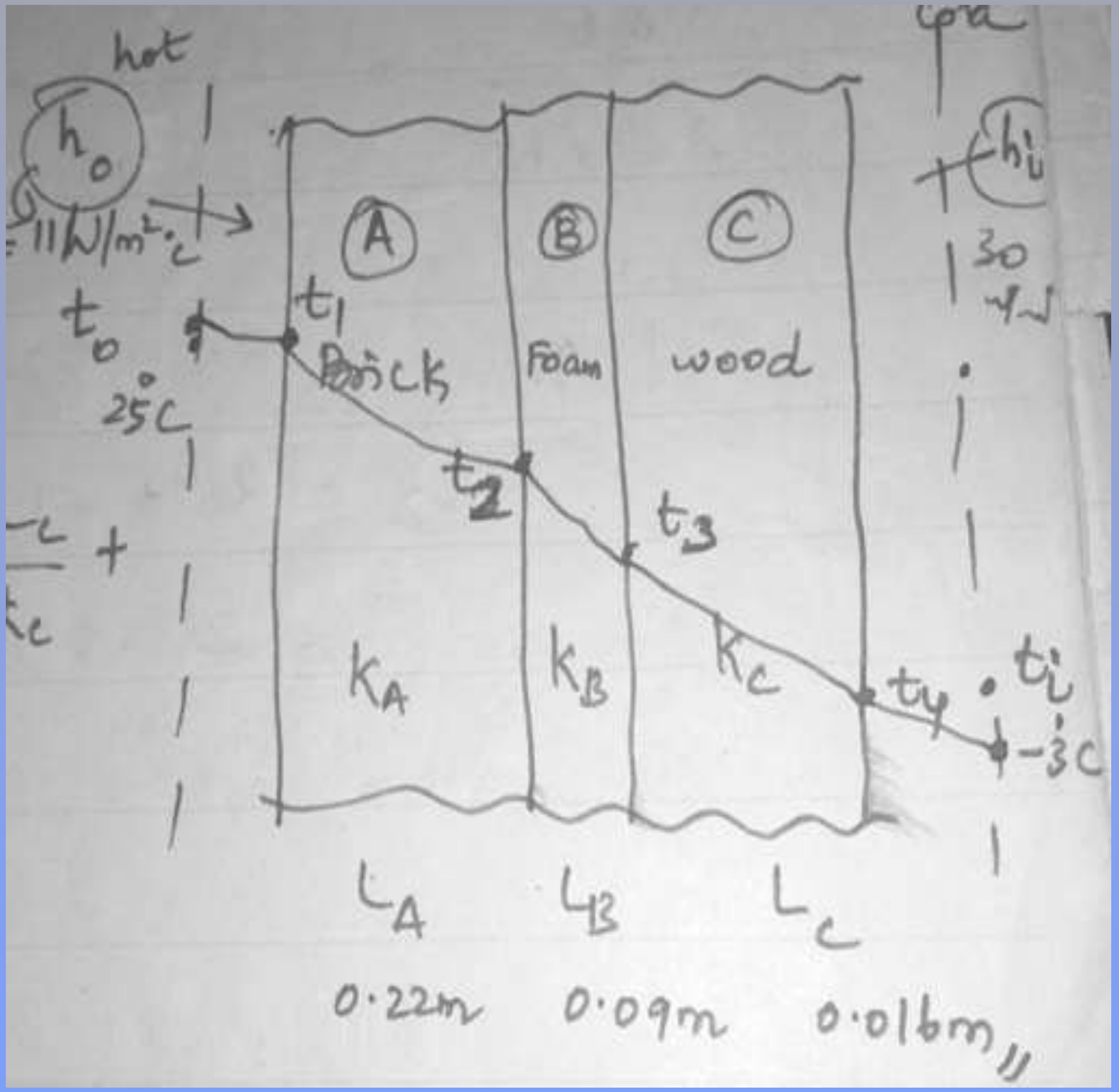
Solution:

$$\frac{1}{U} = \frac{1}{h_i} + \frac{L_A}{K_A} + \frac{L_B}{K_B} + \frac{L_c}{K_c} + \frac{1}{h_o}$$

$$U = 0.2207\ W/m^2\ {}^oC$$

$$Q = 0.2207 \times 85\ (25 - (-3))$$

$$Q = 525.26\ W$$



$$\frac{1}{U} = \frac{1}{h_o} + \frac{L_A}{K_A}$$

$$\frac{1}{11} + \frac{0.22}{0.99}$$

$$= 0.09 + 0.22$$

$$\frac{1}{U} = 0.312$$

→

$$U = 3.2\ W/m^2\ {}^oC$$

$$Q = 3.2 \times 85\,(25 - t_2)$$

$$\frac{526}{3.2 \times 85} = (25 - t_2)$$

$$1.93 = 25 - t_2$$

$$t_2 = 25 - 1.93$$

$$t_2 = 23.07\,^{o}C$$

FINS

FINS:

Heat transfer by convection between a surface and the fluid surrounding it can be increased by attaching to the surface thin strips of metal called fins.

Fins increases the effective area of the surface thereby increasing the heat transfer by convection.

Common types of fin:

1. Infinitely long fin
2. Short fin (end insulated)
3. Short fin (end not insulated)

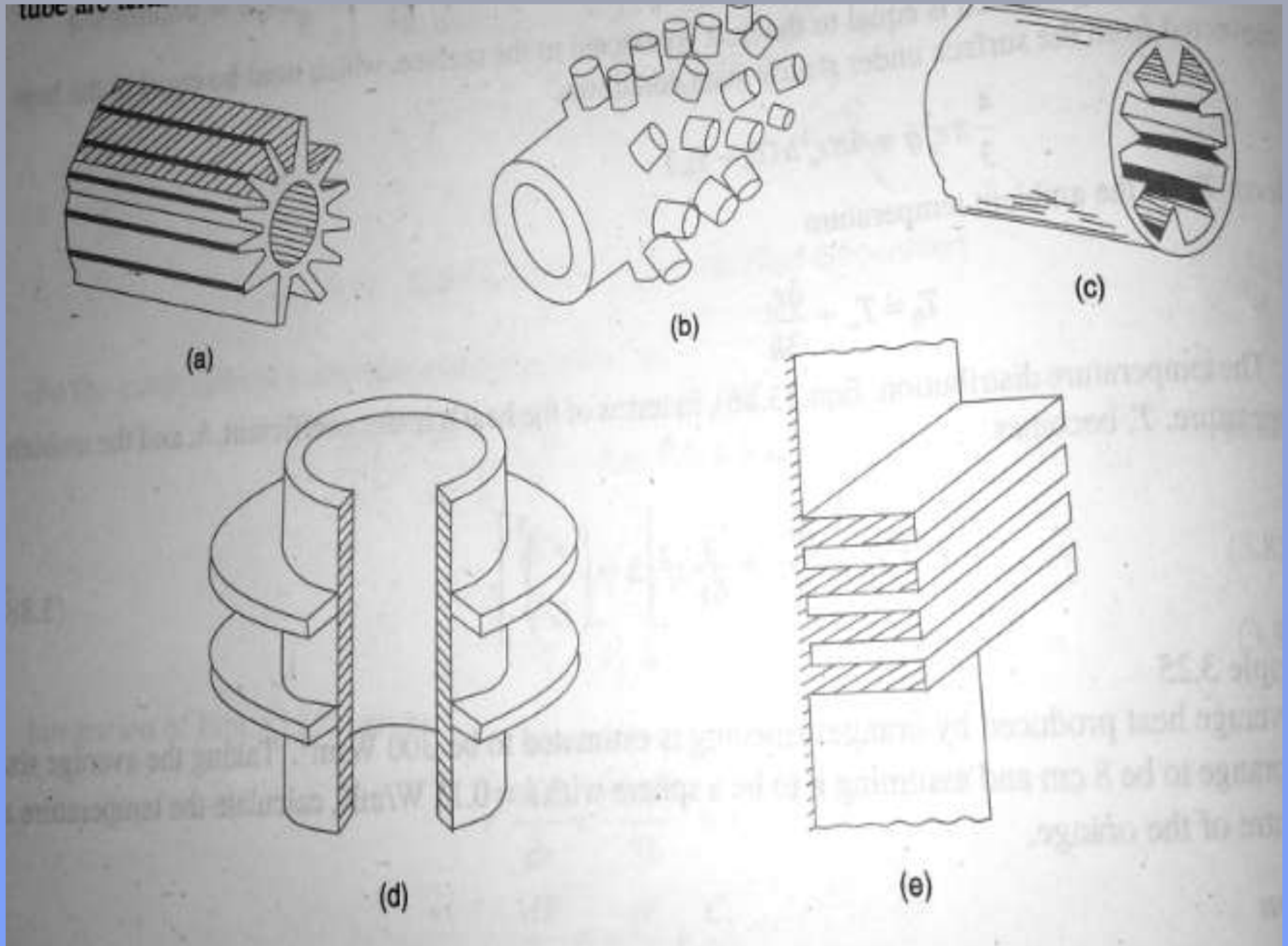


Fig. 3.12 Finned Surfaces: (*a*) External Longitudinal Fins, (*b*) Pin Fins or Spines, (*c*) Internal Longitudinal Fins, (*d*) Circular or Annular Fins, (*e*) Plate Fins.

Temperature distribution and heat dissipation in fin:

A steady state conditions heat balance equation for the small element can be given as

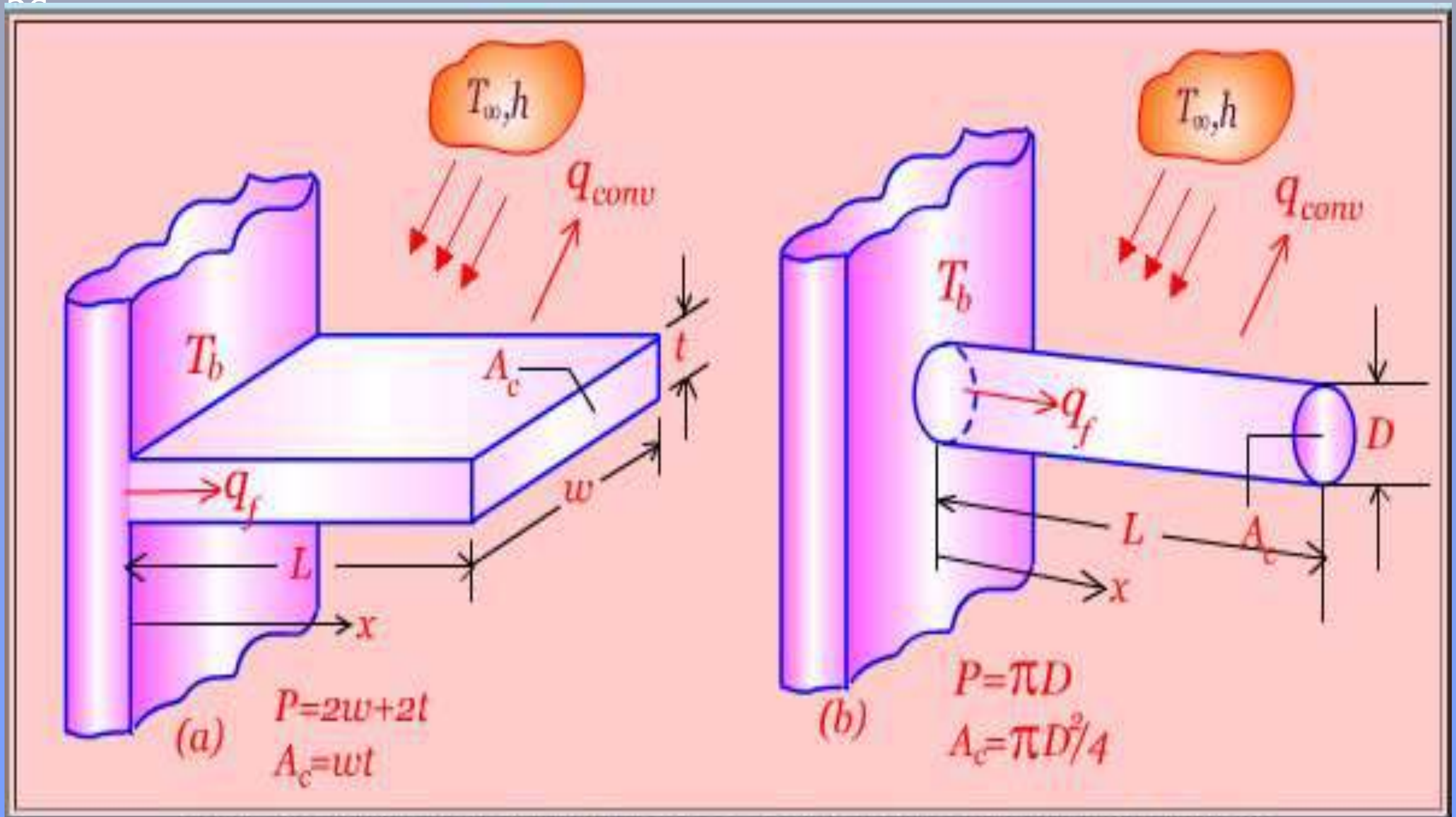


Figure 3.6 Straight Fins Of Uniform Cross Section (a) Rectangular Fin (b) Pin Fin

Heat conducted into the element	=	heat conducted out of the element	+	heat convected to the surrounding air

$$Q_x = Q_{x+dx} + Q_{conv} \quad .....(A)$$

where;

$$Q_x = -k.A.\left(\frac{dt}{dx}\right) \quad ...(1)$$

$$Q_{x+dx} = -k.A.\left(\frac{dt}{dx}\right) - k.A.\left(\frac{d^2t}{dx^2}\right)dx \quad ....(2)$$

$$Q_{x+dx} = h.A.(T - T_\infty) \qquad A = P.dx$$

$$= h.(P.dx).(T - T_\infty) \quad ....(3)$$

Sub (1), (2), (3) in (A)

$$-k.A.\left(\frac{dt}{dx}\right) = -k.A.\left(\frac{dt}{dx}\right) - k.A.\left(\frac{d^2t}{dx^2}\right)dx + h..(P.dx).(T - T_\infty)$$

$$k.A.\left(\frac{d^2t}{dx^2}\right)dx = h..(P.dx).(T - T_\infty)$$

$$k.A.\left(\frac{d^2t}{dx^2}\right) = h..(P).(T - T_\infty)$$

$$\left(\frac{d^2t}{dx^2}\right) = \frac{hP}{k.A.} \quad (T - T_\infty)$$

$$\left(\frac{d^2t}{dx^2}\right) - \frac{hP}{k.A.} \quad (T - T_\infty) = 0$$

$$\left(\frac{d^2t}{dx^2}\right) - m^2 (T - T_\infty) = 0$$ .....(B)

Where;

(D.B. Pg: 50)

$$m^2 = \frac{hP}{k.A.}$$

When $\theta = T - T_\infty$

Equation (B) becomes

$$\left(\frac{d^2t}{dx^2}\right) - m^2\,\theta = 0 \quad \text{..(C)}$$

Equation (C) shows that the temperature is a function of x and m. since it is a 2nd order linear differential equation. The general solution for equation (C) is

$$\theta = C_1\, e^{-mx} + C_2\, e^{mx} \quad \text{.....(D)}$$

The temperature distribution and heat dissipation depends upon the following fin conditions:

Case (i): infinitely long fins

if a fin is infinitely long, the surrounding fluid temperature and the temperature of the fin at its ends are equal.

(i.e.) at

$x = 0;\ T = T_b$

$x = \infty;\ T = T_\infty$

From equation (D)

$$\theta = C_1 e^{-mx} + C_2 e^{mx}$$

Therefore,

$$T - T_\infty = C_1 e^{-mx} + C_2 e^{mx} \quad \text{.....(4)}$$

Substituting the values at $x = 0$; $T = T_b$ In equation (4)

$$(T_b - T_\infty) = C_1 + C_2 \quad \text{......(5)}$$

Substituting the values at $x = \infty$; $T = T_{\infty}$ In equation (4)

$$(T_{\infty} - T_{\infty}) = C_1 e^{-m\infty} + C_2 e^{m\infty}$$

since, $e^{-m\infty} = 0$

$$C_2 e^{m\infty} = 0$$

$e^{m\infty} \neq 0$ → $C_2 = 0$

Sub $C_2 = 0$ in eq (5)

$$T_b - T_{\infty} = C_1 + 0$$

Sub the values of $C_1 + C_2$ in equation (4)

$$(T - T_\infty) = (T_b - T_\infty)\, e^{-mx} + 0$$

$$\frac{(T - T_\infty)}{(T_b - T_\infty)} = e^{-mx}$$

Where;

T = Intermediate temperature in ‘K’

T_b = base temperature in ‘K’

T_∞ = surrounding temperature in ‘K’

x = distance

m =

After knowing the temperature distribution, the heat flow through the fin is obtained by integrating the heat lost by convection over the entire fin surface.

Heat lost by convection

$$Q_{conv} = h.A.\,(T_b - T_\infty)$$

$$= h.(P.dx).\,(T_b - T_\infty)$$

$$\int_0^\infty h.P.\,(T_b - T_\infty).dx$$

$$\int_0^\infty h.P.\,(T_b - T_\infty)\;e^{-mx}.dx$$

since;

$$\frac{(T - T_\infty)}{(T_b - T_\infty)} = e^{-mx}$$

$$Q = h.P.(T_b - T_\infty)\int_0^{\infty} e^{-mx}.dx$$

$$h.P.(T_b - T_\infty)\frac{-1}{m}\left[e^{-mx}\right]_1^{\infty}$$

$$Q = \frac{1}{\sqrt{\frac{kP}{hA_c}}} h.P.(T_b - T_\infty)$$

$$= \sqrt{hPkA}\,(T_b - T_\infty)$$ ...... (From D.B. Pg: 50)

Case (ii): Fin with Insulated End (Short Fins)

This fin has a finite length and the tip of fin is insulated.

at

$x = 0; T = T_b$

$x=L, dT/dx=0$

From equation (4)

.....(4)

$$T - T_\infty = C_1 e^{-mx} + C_2 e^{mx}$$

Differentiate the above equation

$$\left[\frac{dt}{dx}\right] = C_1 e^{-mx}(-m) + C_2 e^{mx}(m) \quad \text{...(a)}$$

Applying the 1st boundary condition´s

x = 0; T = T_b in equation (4)

$$(T_b - T_\infty) = C_1 + C_2 \quad \text{.(b)}$$

2nd boundary condition's:

$x = L;\ \left(\frac{dt}{dx}\right) = 0$ in (a)

$$0 = C_1 e^{-mL}(-m) + C_2 e^{mL}(m)$$

$$= -m.C_1 e^{-mL} + m.C_2 e^{mL}$$

$$m.C_1 e^{-mL} = m.C_2 e^{mL}$$

$$C_1 e^{-mL} = C_2 e^{mL}$$

$$C_1 = C_2 \frac{e^{mL}}{e^{-mL}}$$

$$C_1 = C_2 e^{2mL} \quad \text{......(c)}$$

Substitute the equation (c) in equation (b)

$$T_b - T_\infty = C_2\, e^{2mL} + C_2$$

$$T_b - T_\infty = C_2 \left[e^{2mL} + 1 \right]$$

$$\frac{T_b - T_\infty}{\left[e^{2mL} + 1 \right]} = C_2$$

Sub the value of in C_2 equation (c):

$$C_1 = \frac{T_b - T_\infty}{\left[e^{2mL} + 1\right]} \times e^{2mL}$$

$$C_1 = \frac{T_b - T_\infty}{\left[e^{2mL} + 1\right]} \times \frac{1}{e^{-2mL}}$$

$$C_1 = \frac{T_b - T_\infty}{1 + e^{-2mL}}$$

Substitute the value of C_1 & C_2 in eq (4)

$$T - T_\infty = C_1 e^{-mx} + C_2 e^{mx}$$

$$(T - T_\infty) = \frac{T_b - T_\infty}{1 + e^{-2mL}} e^{-mx} + \frac{T_b - T_\infty}{1 + e^{2mL}} e^{mx}$$

$$(T - T_\infty) = T_b - T_\infty \left[\frac{e^{-mx}}{1 + e^{-2mL}} + \frac{e^{mx}}{1 + e^{2mL}} \right]$$

$$\frac{(T - T_\infty)}{T_b - T_\infty} = \left[\frac{e^{-mx}}{1 + e^{-2mL}} + \frac{e^{mx}}{1 + e^{2mL}} \right]$$

Multiply the numerator & denominator of RHS by e^{-mL} & e^{mL}

$$\frac{(T - T_\infty)}{T_b - T_\infty} = \frac{e^{-mx}}{1 + e^{-2mL}} \times \frac{e^{mx}}{e^{mx}} + \frac{e^{mx}}{1 + e^{2mL}} \times \frac{e^{-mx}}{e^{-mx}}$$

$$\frac{(T - T_\infty)}{T_b - T_\infty} = \frac{e^{m(L-x)} + e^{-m(L+x)}}{e^{mL} + e^{-mL}}$$

In terms of hyperbolic function it can be written as

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

Temperature distribution of fin with insulated end

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m\,(L-x)}{\cosh mL}$$

$$T - T_\infty = T_b - T_\infty \left[\frac{\cosh m\,(L-x)}{\cosh mL} \right]$$

$$\frac{dt}{dx} = T_b - T_\infty \, (-m) \, \frac{\sinh m\,(L-x)}{\cosh mL}$$

We know that

$$Q = -k.A. \left[\frac{dt}{dx}\right]$$

$$Q = -k.A. \left(T_b - T_\infty\right)(-m) \, \frac{\sinh m\,(L-x)}{\cosh mL}$$

$$Q = m\,k.A.\,(T_b - T_\infty)\,\frac{\sinh m\,(L-x)}{\cosh mL}$$

At x = 0;

$$Q = m\,k.A.\,(T_b - T_\infty)\,\frac{\sinh m\,(L-x)}{\cosh mL}$$

$$Q = m\,k.A.\,(T_b - T_\infty)\,\tanh (mL)$$

$$Q_{\text{insulated fin}} = \sqrt{hPkA}\,(T_b - T_\infty)\,\text{Tanh}\;mL$$

Fin Efficiency

Fin Efficiency

The fin efficiency is defined as the ratio of the energy transferred through a real fin to that transferred through an ideal fin. An ideal fin is thought to be one made of a perfect or infinite conductor material. A perfect conductor has an infinite thermal conductivity so that the entire fin is at the base material temperature.

$$\eta = \frac{q_{real}}{q_{ideal}} = \frac{\sqrt{h \cdot P \cdot k \cdot A_c} \cdot \theta_L \cdot \tanh(m \cdot L)}{h \cdot (P \cdot L) \cdot \theta_L}$$

Fin Effectiveness

A fin can effectively enhance heat transfer which is characterized by the fin effectiveness, ϵ_f ,which is as the ratio of fin heat transfer and the heat transfer without the fin. For an adiabatic fin:

$$= \sqrt{\frac{kP}{hA_c}} \tanh(mL)$$

1. INFINITELY LONG FIN:

a) Temperature distribution:

$$\frac{T - T_\infty}{T_b - T_\infty} = e^{-mx}$$

b)

$$Q = \sqrt{hPkA}\,(T_b - T_\infty)$$

2. SHORT FIN:

$$\frac{L}{D} < 30$$

a) Temperature distribution:

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m (L-x)}{\cosh mL}$$

b)

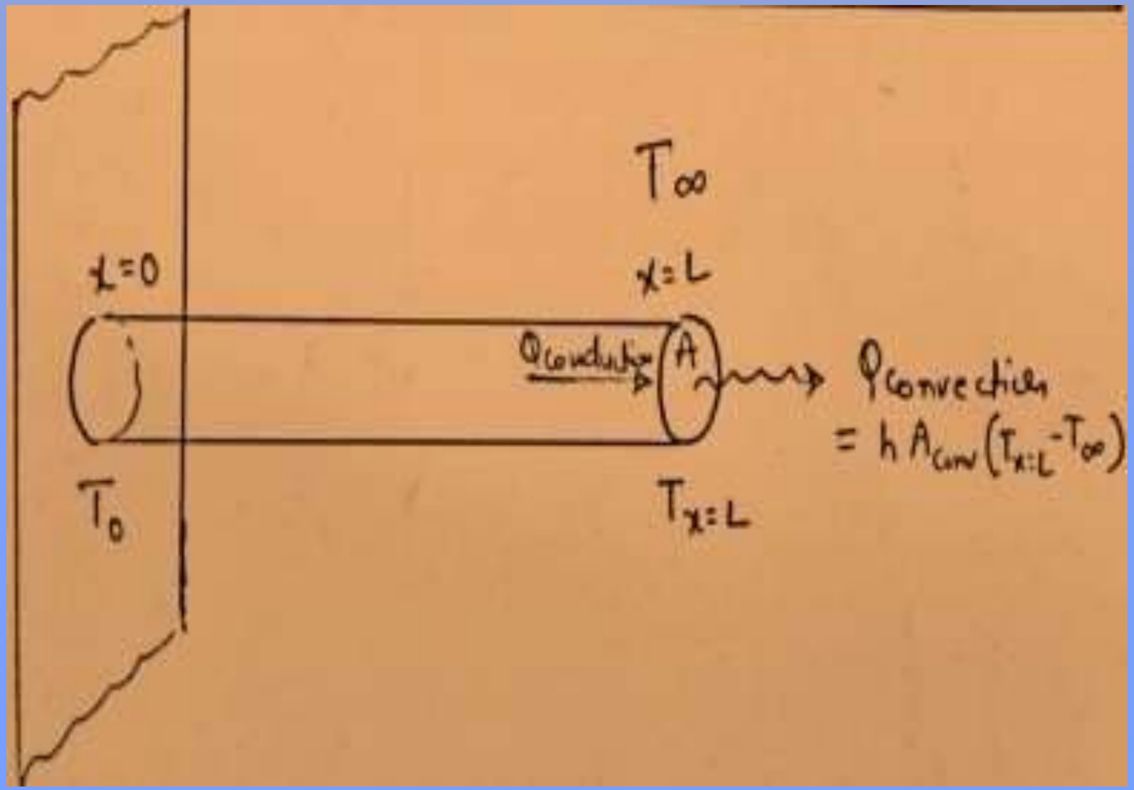
$$Q = \sqrt{hPkA}\,(T_b - T_\infty)\ \text{Tanh}\ mL$$

Case (iii): **SHORT FIN END NOT INSULATED:**

The boundary conditions are:

i. At x = 0, $\theta = \theta_0$

ii. Heat conducted to the fin at x = l = heat convected from the end to surroundings.



$$-k.A.\left(\frac{dt}{dx}\right) = h.A.(T - T_\infty)$$

Where A_{CS} (cross sectin of heat conduction) equals A_{SU} (surface area from which the convection heat transfer takes place), at the tip of the fins

$$A_{CS} = A_{SU}$$

$$\frac{dt}{dx} = \frac{h}{k.}\theta$$ ....(1) at x = l ($\theta = T - T_\infty$)

Applying the boundary condition to the equation

$$T - T_\infty = C_1 e^{mx} + C_2 e^{-mx}$$ ...(A)

At x = 0, $\theta = \theta_o$ we get,

$$\theta = C_1 + C_2$$ (2)

Differentiate equation (A) w.r.t.x.

$$\frac{dt}{dx} = m\,C_1 e^{mx} + m\,C_2\,e^{-mx}$$

$$\left(\frac{dt}{dx}\right)_{x=1} = m\,C_1 e^{\,mL} - m\,C_2\,e^{-mL} \quad \text{.....(3)}$$

Equating 1 & 3

$$\frac{h}{k.}\theta = m\,C_1 e^{\,mL} - m\,C_2\,e^{-mL}$$

$$\frac{h\theta}{k.m} = C_1 e^{mL} - C_2 e^{-mL}$$

$$C_1 e^{mL} - C_2 e^{-mL} = \frac{h}{k.m} \left(C_1 e^{mL} - C_2 e^{-mL} \right) \qquad \text{...(4)}$$

Considering the equations 2 & 4 and solving

$$C_2 = \theta_0 - C_1$$

$$C_1 e^{mL} - (\theta_o - C_1)\, e^{-mL} = -\frac{h}{k.m}\left[C_1 e^{mL} + (\theta_o - C_1)\, e^{-mL}\right]$$

$$C_1 e^{mL} - \theta_o e^{-mL} + C_1 e^{-mL} = -\frac{h}{k.m} C_1 e^{mL} - \frac{h}{k.m} \theta_o e^{-mL} + \frac{h}{k.m} C_1 e^{-mL}$$

$$C_1 = \frac{\theta_o \left[1 - \frac{h}{k.m}\theta_o\right] e^{-mL}}{\left[e^{mL} + e^{-mL}\right] + \frac{h}{k.m}\left[e^{mL} - e^{-mL}\right]}$$

We know

$$C_2 = \theta_o - C_1$$

$$C_2 = \theta_o - \left[\frac{\theta_o \left[1 - \frac{h}{k.m} \theta_o \right] e^{-mL}}{\left[e^{mL} + e^{-mL} \right] + \frac{h}{k.m} \left[e^{mL} - e^{-mL} \right]} \right]$$

$$C_2 = \theta_o \left[1 - \frac{\theta_o \left[1 - \frac{h}{k.m} \theta_o \right] e^{-mL}}{\left[e^{mL} + e^{-mL} \right] + \frac{h}{k.m} \left[e^{mL} - e^{-mL} \right]} \right]$$

$$C_2 = \theta_o \left[1 - \frac{\theta_o \left[1 - \frac{h}{k.m} \theta_o \right] e^{-mL}}{\left[e^{mL} + e^{-mL} \right] + \frac{h}{k.m} \left[e^{mL} - e^{-mL} \right]} \right]$$

$$C_2 = \left[\frac{\theta_o \left[1 + \frac{h}{k.m} \theta_o \right] e^{mL}}{\left[e^{mL} + e^{-mL} \right] + \frac{h}{k.m} \left[e^{mL} - e^{-mL} \right]} \right]$$

Substitute the values of constant C_1 & C_2 in equation (4)

$$\theta = \left[\frac{\theta_o \left[1 - \frac{h}{k.m}\theta_o\right] e^{-mL}}{\left[e^{mL} + e^{-mL}\right] + \frac{h}{k.m}\left[e^{mL} - e^{-mL}\right]} \right] e^{mx}$$

$$+ \left[\frac{\theta_o \left[1 + \frac{h}{k.m}\theta_o\right] e^{mL}}{\left[e^{mL} + e^{-mL}\right] + \frac{h}{k.m}\left[e^{mL} - e^{-mL}\right]} \right] e^{-mx}$$

Take out θ_o

$$\frac{\theta}{\theta_o} = \frac{e^{m(L-x)} + e^{m(L+x)} + \frac{h}{k.m}\left[e^{m(L-x)} - e^{m(L+x)}\right]}{\left[e^{mL} + e^{-mL}\right] + \frac{h}{k.m}\left[e^{mL} - e^{-mL}\right]}$$

Temperature distribution:

$$\frac{\theta}{\theta_o} = \frac{(T - T_\infty)}{T_b - T_\infty} = \frac{\cosh m(L-x) + \frac{h}{k.m} \sinh m(L-x)}{\cosh mL + \frac{h}{k.m} \sinh m(L)}$$

The rate of heat flow from the fin

$$Q_{fin} = -k.A_{cs}\left(\frac{dt}{dx}\right)_{x=0} \quad ...(5)$$

$$(T - T_\infty) = T_b - T_\infty \left[\frac{\cosh m(L-x) + \frac{h}{k.m}\sinh m(L-x)}{\cosh mL + \frac{h}{k.m}\sinh m(L)} \right]$$

Differentiating the expression 'B' we get

$$\left(\frac{dt}{dx}\right)_{x=0} = T_b - T_\infty \left[\frac{-m.\sinh m(L-x) - m\left[\frac{h}{k.m}\cosh m(L-x)\right]}{\cosh mL + \frac{h}{k.m}\sinh m(L)} \right]$$

$$\left(\frac{dt}{dx}\right)_{x=0} = T_b - T_\infty \; (-m) \left[\frac{\sinh mL + \left[\frac{h}{k.m} \cosh mL\right]}{\cosh mL + \frac{h}{k.m} \sinh mL} \right]$$

(6)

Sub 6 in 5

$$Q_{fin} = -k.A_{cs}\,(T_b - T_\infty)\,(-m)\left[\frac{\sinh mL + \left(\frac{h}{k.m}\right)\cosh mL}{\cosh mL + \frac{h}{k.m}\sinh mL}\right]$$

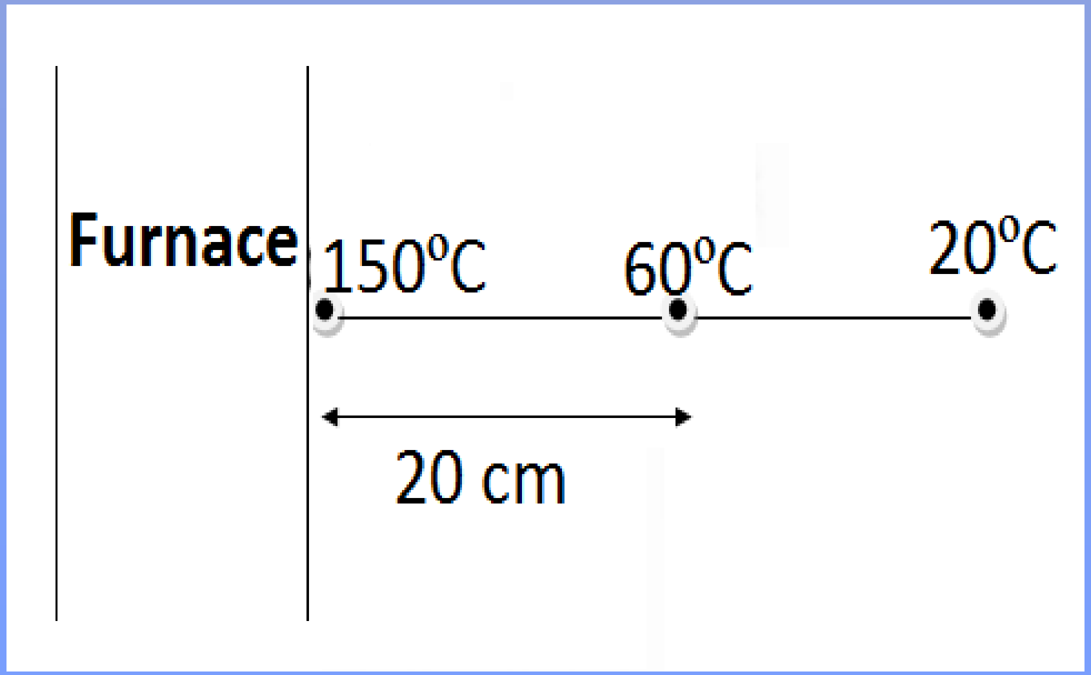
Where;

$$m = \sqrt{\frac{hP}{kA}}$$

$$Q_{fin} = \sqrt{hPkA}\,(T_b - T_\infty)\left[\frac{\tanh mL + \left[\dfrac{h}{k.m}\right]}{1 + \dfrac{h}{k.m}\tanh mL}\right]$$

PROBLEM:

A long rod 5 cm diameter its base is connected to a furnace wall at 150 °C, while the end is projecting into the room at 20 °C. The temperature of the rod at distance of 20 cm apart from its base is 60 °C. The conductivity of the material is 200 W/mK. Determine convective heat transfer coefficient.



Solution:

Condition: fin is a long fin

Refer HMT D.B. Pg: 50

$$\frac{T - T_\infty}{T_b - T_\infty} = e^{-mx}$$

$$\frac{333 - 293}{423 - 293} = e^{-m(0.2)}$$

$$\ln(0.307) = -m \times 0.2 \rightarrow m = 5.9$$

$$m = \sqrt{\frac{hP}{kA}}$$

$$\frac{h \times 0.157}{200 \times 1.96 \times 10^{-3}}$$

$$\frac{(5.9)^2 \times 200 \times 1.96 \times 10^{-3}}{0.157}$$

$$h = 86.9\ W/m^2K$$

An aluminum alloy fin of 7mm thick and 50 mm long protrudes from a wall, which is maintained at 120 ºC. The ambient air temperature is 22 ºC. The heat transfer coefficient and conductivity of the fin material are 140 $W/m^2 K$ and 55 $W/m^2 K$ respectively. Determine:

I. Temperature at the end of the fin

II. Temperature at the middle of the fin

III. Total heat dissipated by the fin

(assume end is insulated)

Solution:

Since the length of the fin is 50 mm, the given problem is treated as short fin problem. Assume the end as insulated.

From HMT D.B Pg: 49,.

Temperature distribution for short fin end insulated:

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m\,(L-x)}{\cosh mL}$$

(1) Temperature at the fin (i.e. x = L)

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m\,(L-x)}{\cosh mL}$$

$$m = \sqrt{\frac{hP}{kA}}$$

$$\frac{140 \times 0.1}{55 \times 3.5 \times 10^{-3}}$$

$$m = 26.9$$

$$P = 2 \times L$$

$$= 2 \times 0.050$$

$$P = 0.1 \text{ m}$$

$$A = L \times t = 0.050 \times 0.007$$

$$A = 3.5 \times 10^{-4} \text{ m}^2$$

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

$$\frac{T - 295}{393 - 295} = \frac{1}{\cosh(26.9 \times 0.050)}$$

$$\frac{T - 295}{393 - 295} = \frac{1}{2.049} \Rightarrow T = 342.8\ K$$

(2) Temperature at the middle of the fin:

(put x = L/2)

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m (L - L/2)}{\cosh mL}$$

$$\frac{T - 295}{393 - 295} = \frac{\cosh m (0.050 - 0.050 / 2)}{\cosh (26.9 \times 0.050)}$$

$$T = 354 \text{ K}$$

(3) Total heat dissipated:

From HMT D.B. Pg: 50

$$= \sqrt{hPkA}\ (T_b - T_\infty)\ \text{Tanh}\ mL$$

$$= (140 \times 0.1 \times 55 \times 3.5 \times 10^{-4})^{1/2} \times (393 - 293) \times \tanh(26.9 \times 0.050)$$

$$= 44\ W$$

A rectangular aluminum fins of 0.5 mm square and 12 mm long are attached on a plane plate which is maintained at 80 °C. Surrounding air temperature is 22 °C. Calculate the no of fins required to generate 35×10^{-3} of heat. Take

$k = 165$ W/mK

$h = 10$ W/m^{2}K

Assume no heat loss from the tip of the fin.

Solution:

Since the problem is associated with short fin with end insulated.

$$\text{No. of fins required} = \frac{\text{Heat generated}}{\text{Heat transfer / fin}}$$

Heat generated $= 35 \times 10^{-3}$ W

Heat transfer / fin $= \sqrt{hPkA}\ (T_b - T_\infty)$ Tanh mL

$P = 2\ (b + t)$

$= 2\ [\ 0.5 \times 10^{3} + 0.5 \times 10^{3}]$

$P = 2 \times 10^{3}$ m

$A = b \times t = 0.5 \times 10^{3} \times 0.5 \times 10^{3}$

$A = 2.5 \times 10^{-7}\ m^2$

$$m = \sqrt{\frac{hP}{kA}}$$

$$\sqrt{\frac{10 \times 2 \times 10^{-3}}{165 \times 2.5 \times 10^{-3}}}$$

$$m = 22\ m^{-1}$$

Heat transfer / fin:

$$= \sqrt{hPkA}\ (T_b - T_\infty)\ \text{Tanh}\ mL$$

$$= (10 \times 2 \times 10^{-3} \times 165 \times 2.5 \times 10^{-7})^{1/2} \times (353 - 295) \times \tanh(22 \times 12 \times 10^{-3})$$

$$= 0.0135\ W / \text{fin}$$

$$\text{No. of fins required} = \frac{\text{Heat generated}}{\text{Heat transfer / fin}}$$

$$= \frac{35 \times 10^{-3}}{0.0135}$$

No. of fins required = 2.59 = 3

A circumferential rectangular fins of 140 mm wide and 5mm thick are fitted on a 200 mm diameter tube. The fin base temperature is 170 ºC and the ambient temperature is 25 ºC. Estimate fin efficiency and heat loss per fin.

Take

k = 200 W/mK

h = 140 W/m ^{2}K

Solution:

From HMT D.B. Pg: 51

Converted length =

$$L_c = L + t / 2$$

$$L_c = 0.14 + 0.005 / 2 \rightarrow L_c = 0.1425$$

$$r_{2c} = r_1 + L_c$$

$$= 0.1 + 0.1425$$

$$= 0.2425$$

$$A_m = t\,(r_{2c} - r_1)$$

$$= 0.005\,(0.2425 - 0.100)$$

$$= 0.005\,(0.1425)$$

$$= 7.125 \times 10^{-4}\ m^2$$

$$A_s = 2\prod (r_{2c}^2 - r_1^2)$$

$$= 2\prod (0.2425^2 - 0.1^2)$$

$$= 0.3066\ m^2$$

$$X_{axis} = L_c \left[\frac{h}{kA_m} \right]^{0.5}$$

$$(0.1425)^{1.5} \left[\frac{140}{220 \times 7.125 \times 10^{-4}} \right]^{0.5} \Rightarrow X_{axis} = 1.60$$

$$\text{curve}\left(\frac{r_{2c}}{r_1}\right) = \frac{0.2425}{0.1} = 2.425$$

From D.B Pg: 51

Fin efficiency = 33%

$$Q = \eta A_s h (T_b - T_\infty)$$

$$= 0.3 \times 0.3066 \times 140 (443 - 298)$$

$$= 1867 \text{ W}$$

A stainless steel cylindrical rod fin of 1.2 cm diameter and 6 cm height with thermal conductivity of 25 W/mK is exposed to surrounding with a temperature of 60 ºC, the heat transfer coefficient is 45 W/m^2K and the temperature at the base of the fin is 100 ºC. Determine.

1. Fin efficiency
2. Temperature at the edge of the rod
3. Heat dissipation
4. Fin effectiveness

Solution:

1. Fin efficiency (for insulated ends)

$$\eta_{tip} = \frac{\tanh mL}{mL}$$

1. Fin efficiency (for insulated ends)

$$\eta_{tip} = \frac{\tanh mL}{mL}$$

From HMT D.B Pg. 50

$$m = \sqrt{\frac{hP}{kA}}$$

$$\frac{45 \times 0.0376}{25 \times 1.13 \times 10^{-4}} \Rightarrow m = 24.4\ m^{-1}$$

$$\frac{\tanh(24.4 \text{ x } 6 \text{ x } 10^{-2})}{(24.4 \text{ x } 6 \text{ x } 10^{-2})}$$

$$\eta_{fin} = 0.61 = 61\%$$

(2) Temperature at the edge of the rod:

Temperature distribution: (short fin and insulated)

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

$$\frac{T - 333}{373 - 333} = \frac{1}{\cosh(24.4 \times 6 \times 10^{-2})}$$

$\Rightarrow$ $T = 35\ K$

(3) Heat dissipation for short fin end insulated:

$$= \sqrt{hPkA}\ (T_b - T_\infty)\ \text{Tanh}\ mL$$

$$= (45 \times 0.0376 \times 25 \times 10^{-4})^{1/2} \times (373 - 333) \times \tanh(24.4 \times 6 \times 10^{-2})$$

$$= 2.33\ \text{W}$$

(4) Fin effectiveness (for insulated tip)

$$\eta_{tip} = \frac{\tanh mL}{\sqrt{\frac{hP}{kA}}}$$

$$\frac{\tanh(24.4 \times 6 \times 10^{-2})}{0.0691}$$

$$\epsilon = 12.4$$

TRANSPORT EQUATIONS FOR MOVING-BOUNDARY PROBLEMS

A. The Differential Equations

Consider two media separated at any time t by an interface $S(t)$ whose position in space is an unknown function of time. The two media are bounded in space by the two surfaces S_I and S_{II} whose equations relative to a conveniently chosen three dimensional coordinate system have the form $S_I(\bar{r}) = 0$, $S_{II}(\bar{r}) = 0$. Medium I occupies the region bounded by the surfaces S_I and $S(t)$, medium II occupies the region bounded by $S(t)$ and S_{II} . The interfaces S_I and S_{II} being fixed in space will be called the fixed interfaces. The interface $S(t)$ whose position is a function of time will be called the "moving interface."

$$\nabla \cdot \bar{v} = 0$$

$$\frac{D\bar{v}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 \bar{v}$$

$$\frac{DT}{Dt} = \alpha \nabla^2 T$$

$$\frac{D\rho_A}{Dt} = \mathcal{D} \nabla^2 \rho_A$$

Most physical problems involve some sort of special geometric feature. Linear, cylindrical, and spherical goemetries are typical examples. In these instances, the moving boundary is described goemetrically as a plane, cylindrical, or spherical moving boundary and the problem may be described by only two variables: time and a single position variable. Under these conditions, the transport equations 2.1 through 2.4 reduce to the following

$$\frac{\partial}{\partial r}\left(r^{n-1} v_r\right) = 0 \qquad .5)$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} = \nu\, r^{1-n} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial v_r}{\partial r}\right) - \frac{1}{\rho} \frac{\partial p}{\partial r} \qquad .6)$$

$$\frac{\partial \rho_A}{\partial t} + v_r \frac{\partial \rho_A}{\partial r} = \mathcal{D}\, r^{1-n} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial \rho_A}{\partial r}\right) \qquad .7)$$

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} = \alpha\, r^{1-n} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial T}{\partial r}\right) \qquad (2.8)$$

where r is a general position variable and n takes on the values 1,2,3 corresponding to linear, cylindrical, and spherical geometrics respectively. The equations for the fixed interfaces S_I , S_{II} and the moving interface $S \equiv S(t)$ take respectively the following simple forms

$$r = r_I \ , \ r = r_{II} \ , \ r = S(t)$$

where r_I and r_{II} are known fixed values and $S(t)$ is an unknown function of time.

B. The Initial Conditions

The initial temperature and concentration fields in each medium are specified functions of position and can be generally expressed by the following equations

$$T\ (r,0) = g(r)$$

$$\rho_A(r,0) = h(r)$$

The initial position of the moving interface; that is, $S(0)$ is also specified.

Thank you