CONTINUOUS RANDOM VARIABLE

The density function of a random variable x is given by $f(x) = kx(2-x)$, $0 \le x < 2$. Find k, mean, variance and 3^{rd} moment.

Soln:

(i) To find k:

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{0}^{2} kx(2-x)\,dx = 1$$

$$\Rightarrow k\int_0^2 (2x - x^2)\,dx = 1$$

$$\Rightarrow k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow k\left[4 - \frac{8}{3}\right] = 1$$

$$\Rightarrow k = 3/4$$

$$\therefore f(x) = \tfrac{3}{4}x(2-x)$$

(ii) To find mean:

WKT $E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^2 x \cdot \frac{3}{4} x(2-x) dx = \frac{3}{4} \int_0^2 (2x^2 - x^3) dx$

$$= \frac{3}{4}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = 1$$

(iii) To find variance:

Now $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^2 x^2 \cdot \frac{3}{4} x(2-x) dx = \frac{3}{4} \int_0^2 (2x^3 - x^4) dx$

$$= \frac{3}{4}\left[2\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{6}{5}$$

$\therefore Var(x) = E(x^2) - (E(x))^2 = \frac{6}{5} - 1 = 1/5$

(iv) To find 3rd moment:

$E(x^3) = \int_{-\infty}^{\infty} x^3 f(x) dx = \int_0^2 x^3 \cdot \frac{3}{4} x(2-x) dx = \frac{3}{4} \int_0^2 (2x^4 - x^5) dx$

$$= \frac{3}{4}\left[2\frac{x^5}{5} - \frac{x^6}{6}\right]_0^2 = \frac{8}{5}$$

The p.d.f of a cts. R.V x is $f(x) = ke^{-|x|}$. Find k & $F(x)$

Soln:

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} ke^{-|x|}\,dx = 1$$

$$2k\int_{0}^{\infty} e^{-x}\,dx = 1$$

$$2k \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$2k = 1 \Rightarrow k = 1/2$$

$\therefore f(x) = \frac{1}{2}e^{-|x|}$

Now $f(x) = \frac{1}{2}e^{-|x|} = \begin{cases} \frac{1}{2}e^{-(-x)}, & -\infty < x < 0 \\ \frac{1}{2}e^{-(x)}, & 0 < x < \infty \end{cases}$

$$= \begin{cases} \frac{1}{2}e^{x}, & -\infty < x < 0 \\ \frac{1}{2}e^{-x}, & 0 < x < \infty \end{cases}$$

(ii) For $x \le 0$,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{1}{2}e^{x}\,dx = \frac{1}{2}e^{x}\Big]_{-\infty}^{x} = \frac{e^{x}}{2}$$

For $x > 0$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} \tfrac{1}{2} e^{x}\,dx + \int_{0}^{x} \tfrac{1}{2} e^{-x}\,dx = \frac{1}{2} e^{x}\Big]_{-\infty}^{0} + \frac{1}{2} \frac{e^{-x}}{-1}\Big]_{0}^{x}$$

$$= \frac{1}{2} + \frac{1}{2}\left[e^{-x} + 1\right]$$

$$= \frac{1}{2}\left[2 - e^{-x}\right]$$

A continuous random variable x has the distribution function $F(x) = \begin{cases} 0, & x \le 1 \\ k(x-1)^4, & 1 < x \le 3 \\ 0, & x > 3 \end{cases}$. Find pdf: $f(x)$, k. and $P[X < 2]$.

Soln:

(i) If $F(x)$ given then to find $f(x)$, WKT

$$f(x) = \frac{d}{dx}\left[F(x)\right]$$

$$= \frac{d}{dx}\left[k(x-1)^4\right] = 4k(x-1)^3$$

$$\therefore f(x) = \begin{cases} 0, & x \le 1 \\ 4k(x-1)^3, & 1 < x \le 3 \\ 0, & x > 3 \end{cases}$$

(ii) To find k:

$$\text{WKT} \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\int_{1}^{3} 4k(x-1)^3\,dx = 1$$

$$4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$\Rightarrow k(2^4) - k(0) = 1$$

$$\Rightarrow 16k = 1 \Rightarrow k = 1/16$$

$$\therefore f(x) = \begin{cases} 0 & , x \le 1 \\ \frac{1}{4}(x-1)^3, & 1 < x \le 3 \\ 0 & , x > 3 \end{cases}$$

(iii) $P[X<2]$:

WKT $P[X \le x] = F[x]$

$P[X<2] = F[2]$

$= K[2-1]^4 = \frac{1}{16}$