MOMENTS AND MOMENT GENERATING FUNCTIONS

TO FIND MOMENTS :

$$\mu_1' = E(x) = \sum_{i=1}^{\infty} x_i p(x_i) \text{ , first moment}$$

$$\mu_2' = E(x^2) = \sum_{i=1}^{\infty} x_i^2 p(x_i) \text{ , second moment}$$

$$\mu_3' = E(x^3) = \qquad \text{ , Third moment}$$

$$\mu_4' = E(x^4) = \qquad \text{ , fourth moment.}$$

. . . .

$$\mu_r' = E(x^r) = \sum_{i=1}^{\infty} x_i^r p(x_i) \text{ , } r^{th} \text{ moment .}$$

MOMENT GENERATING FUNCTION [MGF] :

$$M_x(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} p(x)$$

Find the MGF of the RV x whose probability func.

$p(X=x)=\frac{1}{2^x}$, $x=\cancel{0}\,1,2,\ldots$ Find its mean & Variance.

Soln:

$$M_x(t)=E(e^{tx})=\sum_{x=1}^{\infty}e^{tx}p(x)$$

$$=\sum_{x=1}^{\infty}e^{tx}\frac{1}{2^x}=\sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x$$

$$=\frac{e^t}{2}+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\cdots$$

$$=\frac{e^t}{2}\left[1+\frac{e^t}{2}+\left(\frac{e^t}{2}\right)^2+\cdots\right]=\frac{e^t}{2}\left[1-\frac{e^t}{2}\right]^{-1}$$

$$=\frac{e^t}{2}\cdot\frac{2}{2-e^t}=\frac{e^t}{2-e^t}$$

To find mean & variance:

$$\text{Mean } (\mu_1')=\left\{\frac{d}{dt}\left[M_x(t)\right]\right\}_{t=0}$$

$$=\left\{\frac{d}{dt}\left[\frac{e^t}{2-e^t}\right]\right\}_{t=0}=\left\{\frac{d}{dt}\left[e^t(2-e^t)^{-1}\right]\right\}_{t=0}$$

$$=\left[e^t(-1)(2-e^t)^{-2}(-e^t)+(2-e^t)^{-1}e^t\right]_{t=0}$$

$$=(-1)(2-1)^{-2}(-1)+(2-1)^{-1}(1)$$

$$=(1)^{-2}+(1)^{-1}=2$$

Now $\mu_2' = \left\{\dfrac{d^2}{dt^2}\left[M_x(t)\right]\right\}_{t=0}$

$$= \left\{\frac{d^2}{dt^2}\left[\frac{e^t}{2-e^t}\right]\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left[e^{2t}(2-e^t)^{-2} + (2-e^t)^{-1}e^t\right]\right\}_{t=0}$$

$$= \left\{\left[e^{2t}(-2)(2-e^t)^{-3}(-e^t) + (2-e^t)^{-2}e^{2t}\cdot 2\right] + \left[(2-e^t)^{-1}e^t + e^t(-1)(2-e^t)^{-2}(-e^t)\right]\right\}_{t=0}$$

$$= \left\{2e^{3t}(2-e^t)^{-3} + 2e^{2t}(2-e^t)^{-2} + (2-e^t)^{-1}e^t + e^{2t}(2-e^t)^{-2}\right\}_{t=0}$$

$$= 2(2-1)^{-3} + 2(2-1)^{-2} + (2-1)^{-1} + (2-1)^{-2}$$

$$= 2(1)^{-3} + 2(1)^{-2} + (1)^{-1} + (1)^{-2}$$

$$= 4 + 2$$

$\therefore$ Variance $= \mu_2' - \mu_1^2 = 6 - (2)^2 = 2$

Let x be the RV with probability law $p(x=r) = q^{r-1}p$; $r = 1, 2, 3, \ldots$ Find MGF and also mean; variance assuming $p+q=1$

Find MGF of an exponential random variable & hence find the mean & variance.

Soln: For exponential random variable, WKT

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

To find MGF:

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_{0}^{\infty} e^{-(\lambda - t)x}\, dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_{0}^{\infty}$$

$$= \frac{\lambda}{\lambda - t}$$

To find Mean:

$$E(x) = \mu_1' = \left\{\frac{d}{dt}\left[M_x(t)\right]\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right)\right\}_{t=0}$$

$$= \left\{\frac{\lambda}{(\lambda - t)^2}\right\}_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$\therefore \ E(x) = \frac{1}{\lambda}$$

To find Variance:

Now, $E(x^2) = \mu_2' = \left\{\frac{d^2}{dt^2} M_x(t)\right\}_{t=0}$

$$= \left\{\frac{d}{dt} \frac{\lambda}{(\lambda - t)^2}\right\}_{t=0}$$

$$= \left\{\frac{2\lambda}{(\lambda - t)^3}\right\}_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$\therefore \ Var(x) = E(x^2) - (E(x))^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$