Unit Load Method

Proof of Unit Load Method

Consider the body shown in fig. which is subjected to forces $P_1, P_2, P_3, P_4 \ldots \ldots P_n$ applied gradually.

Let the displacement under load points be $\Delta_1, \Delta_2, \Delta_3 \ldots \Delta_n$ and at point 'C' be Δ.

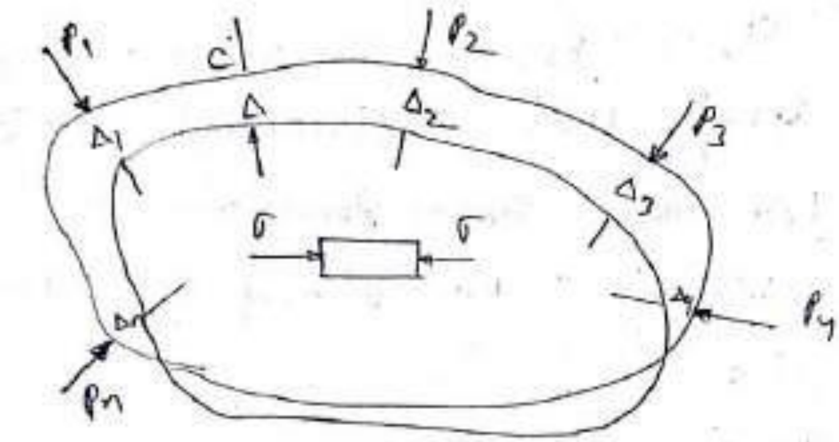


Fig ①

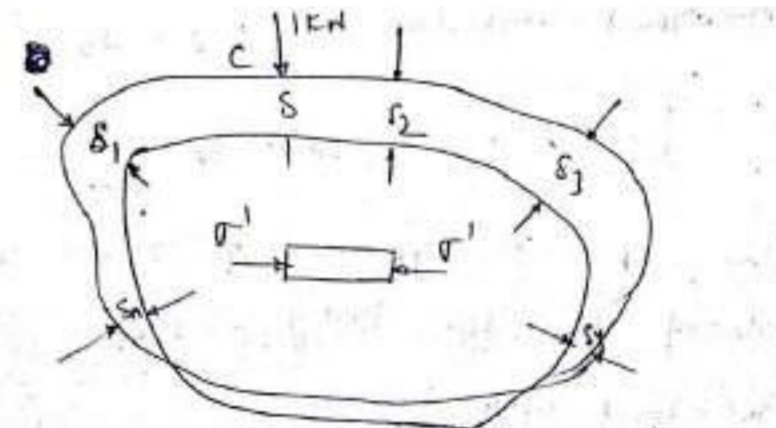


Fig ②

and Strain energy stored = $\int \frac{1}{2} \sigma . e . dv$.

where σ = stress

e = strain in the element.

$\therefore$ External work done = Strain energy stored

$$\frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \frac{1}{2}\Delta_3 P_3 + \cdots + \frac{1}{2}\Delta_n P_n = \int \frac{1}{2}\sigma . e\, dv \quad \text{--- (1)}$$

Now, Consider the same body subjected to an unit load applied gradually at 'c' when it is free of system of 'P' forces.

Let the displacements at 1, 2, 3 … n be $\delta_1, \delta_2, \delta_3, \ldots \delta_n$ respectively and the displacement at 'c' be 'δ'.

Let the stress produced in the element be σ' and the strain be e'.

External work done $= \frac{1}{2} \times 1 \times \delta$.

Internal work done $= \int \frac{1}{2} \sigma' e' \, dv$.

$$\frac{1}{2} \times 1 \times \delta = \int \frac{1}{2} \sigma' e' . dv \quad \text{—— (2)}$$

Now, if 'P' system of forces is applied to the body as shown in fig (2)

External work done

$$= \frac{1}{2} \Delta_1 P_1 + \frac{1}{2} \Delta_2 P_2 + \cdots + \frac{1}{2} \Delta_n P_n + 1 \times \Delta$$

Since, unit load is already acting.

Internal work done $= \int \frac{1}{2}\sigma \cdot e\, dv + \int \sigma' e\, dv$,

Since the stress σ' is acting throughout the deformation.

External work done = Internal work done

$$\frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \cdots + \frac{1}{2}\Delta_n P_n + 1 \times \Delta$$

$$= \int \frac{1}{2}\sigma \cdot e\, dv + \int \sigma' e\, dv. \quad \text{—— (3)}$$

Subtracting equation (1) from (3)

$$1 \times \Delta = \int \sigma' \cdot e \cdot dv$$

or

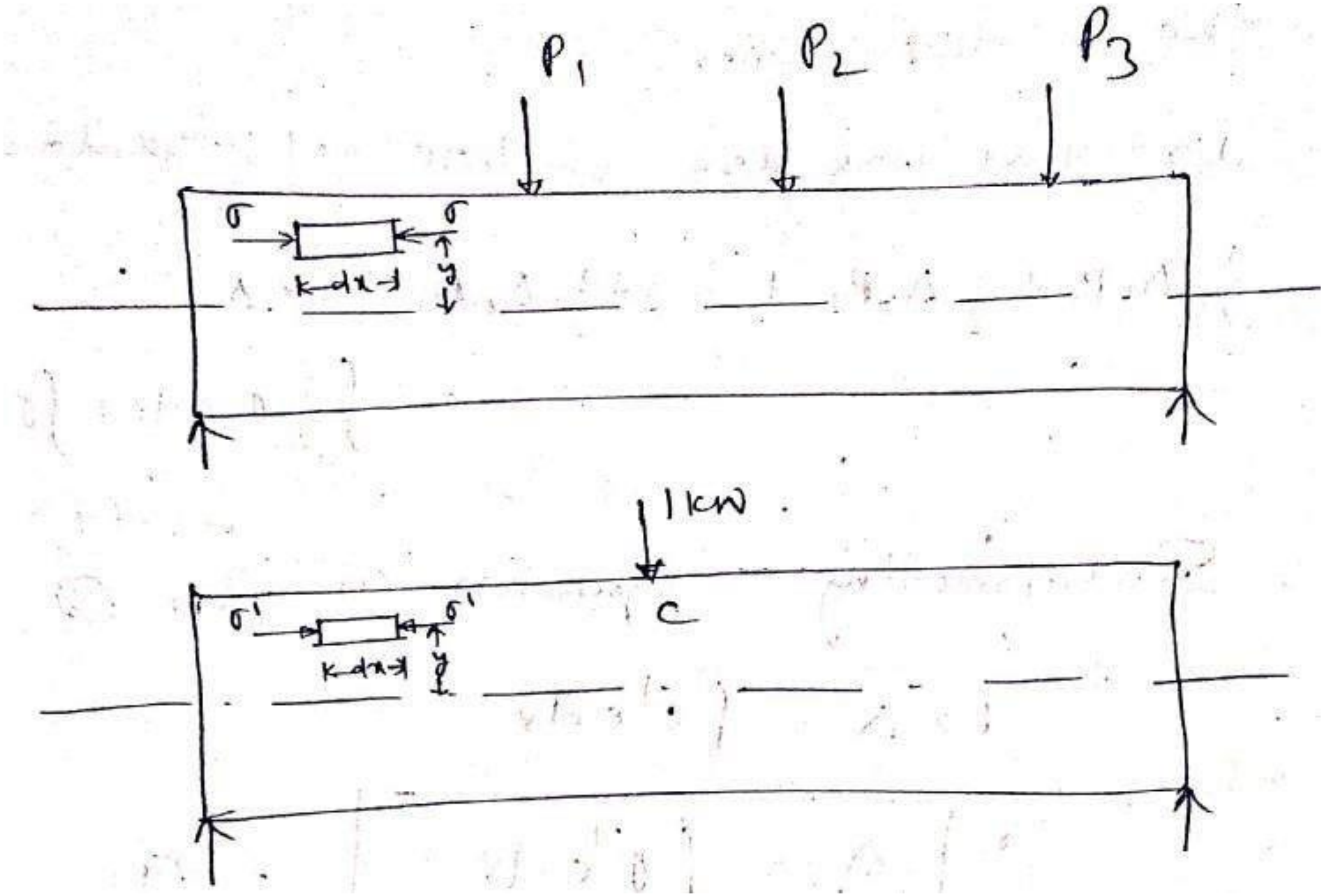
$$\boxed{\Delta = \int \sigma' e\, dv} \quad \text{—— (4)}$$

where Δ = deflection at point where unit load is applied and is measured in the direction of unit load,

σ' = Stress in an element due to unit load.

and e = Strain in the element due to given load system.

Application of Unit Load Method to Beam Deflection



Consider the beam subjected to a system of 'p' forces.

The stress in the element at a distance 'y' from neutral axis is

$$\sigma = \frac{M}{I} y$$

where 'M' = Moment acting at the section

Strain in the element due to given system of forces is

$$e = \frac{M}{EI} y$$

Let 'm' is the moment at the section due to unit load acting at 'c'

Then stress = $\boxed{\sigma' = \frac{M}{I} \cdot y}$

From equation (1)

Deflection = $\boxed{\Delta = \int \sigma' e \, dV}$

Putting the value of σ' & e in the above equation

$$\Delta = \int \frac{m}{I} \cdot y \cdot \frac{M}{EI} y \cdot dV$$

$$\Delta = \int \frac{mM}{EI^2} \cdot y^2 \, dv$$

$$= \int_0^L \frac{Mm}{EI^2} \left[\int_0^A y^2 \, dA \right] dx$$

But $\int_0^A y^2 \, dA = I$

$$\therefore \Delta = \int_0^L \frac{Mm}{EI^2} \cdot I \cdot dx$$

$$\Delta = \int_0^L \frac{Mm}{EI} \, dx \quad \text{--- (5)}$$

The equation (5) used to find out the deflection at any point 'c'. It needs bending moment due to a given load system and unit load acting at 'c'.

This is Procedure is applicable to rigid frames also, where only flexure effect is considered (i.e in the analysis in which the effect of axial & shear forces are neglected)

Deflection by Unit Load Method

Deflection by Unit Load Method

- This method is applicable to beam and rigid frame where only flexural effect is considered.
- In the analysis, the effect of axial force and shear forces are neglected.
- The deflection at any point can be find out by:

$$\Delta = \int_0^L \frac{Mm}{EI} dx$$

Where, M = Bending moment at the section due to the external forces
m = Bending moment at the section due to unit loading
E = Modulus of Elasticity
I = Moment of Inertia of the section

Q1. Determine the deflection at the free end of the over hanging beam shown in Figure by unit load method.

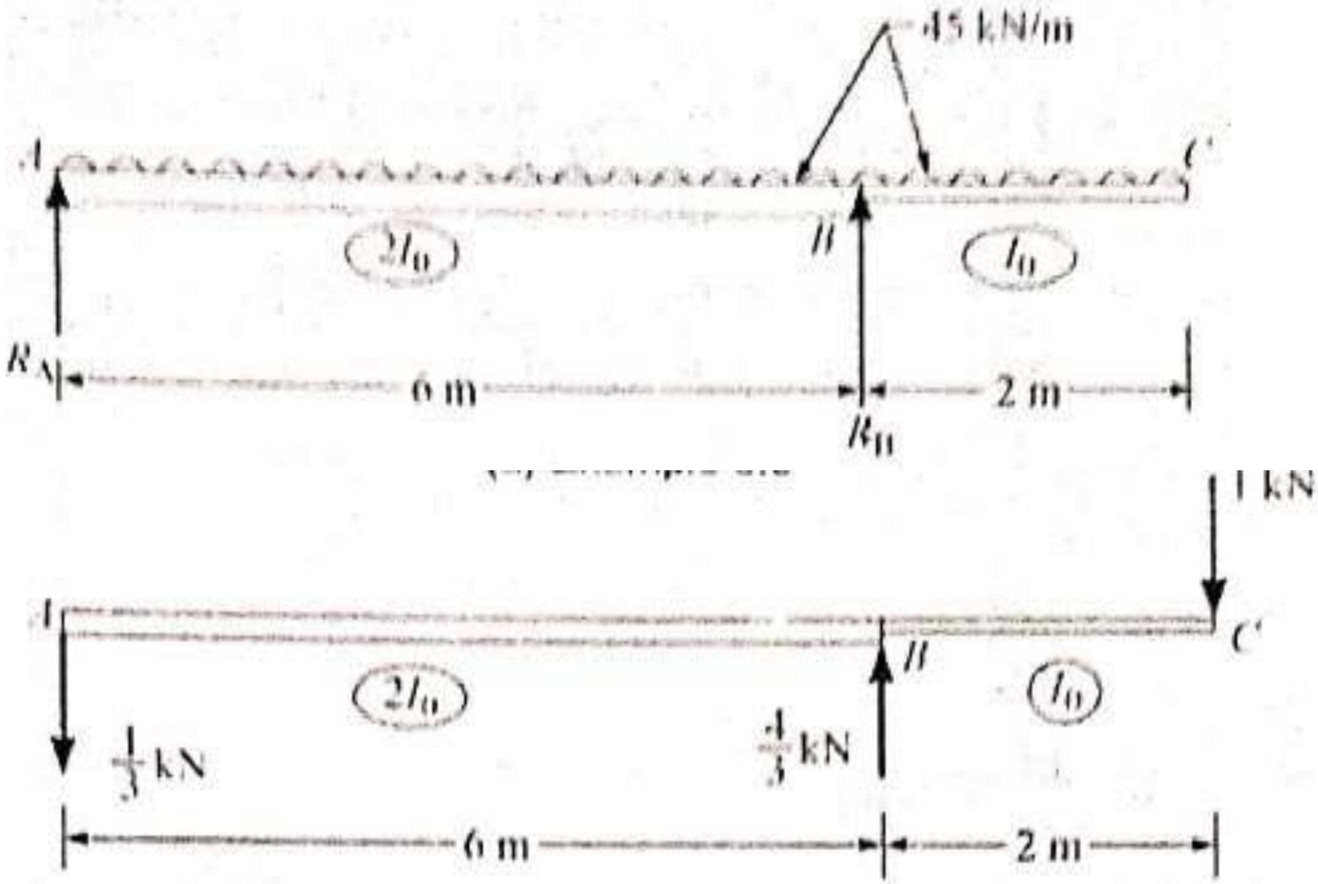


Figure 1: Beam with unit load at ‘*C*’

- Find out the reactions due to external forces, taking moment about A

$$\sum M_A = 0, \text{ gives}$$
$$R_B \times 6 = 45 \times 8 \times 4$$
$$R_B = 240 \text{ kN}$$
$$\sum V = 0, \text{ gives}$$
$$R_A = 45 \times 8 - 240 = 120 \text{ kN}$$

- Find out the reactions, when unit load acting at '*C*'

$$R_B = \frac{1 \times 8}{6} = 1.333 \text{ kN}$$
$$R_A = 0.333 \text{ kN} \downarrow$$

- Taking sagging moment as positive and hogging moment as negative, find out the expressions for moments in various portions of the beam due to external loading and unit force where the deflection is to be determined in a Tabular form.

Portion	AB	BC
Origin	A	C
Limit	0 – 6	0 – 2
M	$120x - \frac{1}{2} \times 45x^2$	$-\frac{1}{2} \times 45x^2$
m	$-0.333x$	$-x$
I	$2I_0$	I_0

$$\Delta_c = \int_0^6 \frac{(120x - 22.5x^2)(-0.333x)dx}{E2I_0} + \int_0^2 \frac{(-22.5x^2)(-x)dx}{EI_0}$$

$$= \int_0^6 \frac{(-20x^2 + 3.7x^3)dx}{EI_0} + \frac{1}{EI_0}\int_0^2 22.5x^3 dx$$

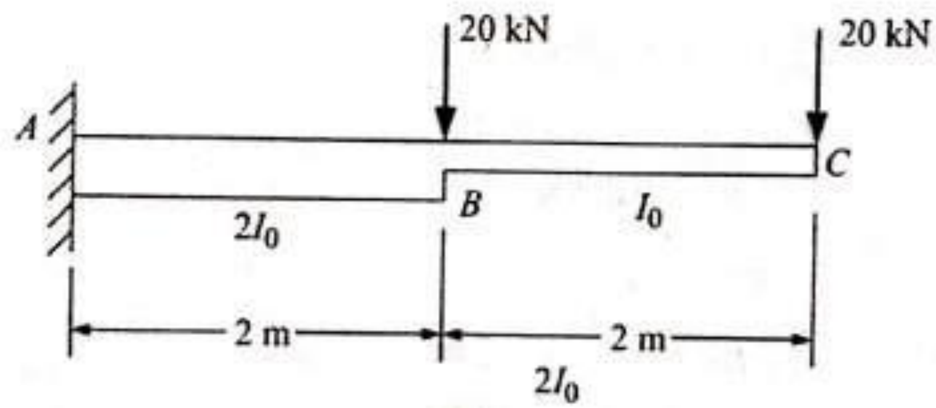
$$= \frac{1}{EI_0}\left[-\frac{20x^3}{3} + \frac{3.75x^4}{4}\right]_0^6 + \frac{1}{EI_0}\left[\frac{22.5x^4}{4}\right]_0^2$$

$$= \frac{1}{EI_0}\left[-\frac{20\times 6^3}{3} + \frac{3.75\times 6^4}{4} + \frac{22.5\times 2^4}{4}\right]$$

$$= -\frac{135}{EI_0}$$

$$= \frac{135}{EI_0}\text{, upward}$$

Q2. Determine the deflection and rotation at the free end of the cantilever beam shown in Figure by unit load method. Given $E = 200000$ N/mm² and $I = 12 \times 10^6$ mm⁴



- Find out the deflection and rotation at the free end of the cantilever beam, apply unit load for deflection and unit moment for rotation at the free end of the beam as shown in Figure.

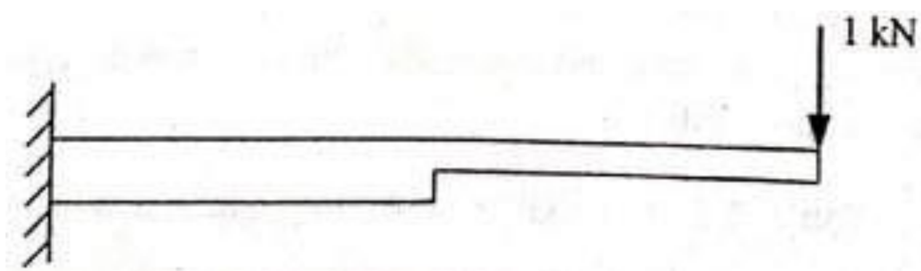


Figure 1: Beam with unit vertical load at '*C*'

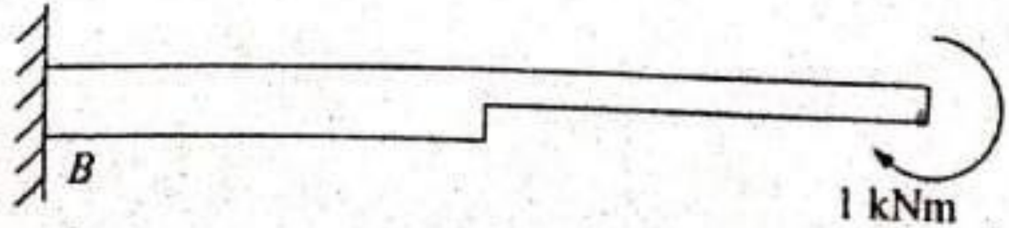


Figure 2: Beam with unit moment at ‘*C*’

- The bending moment expressions can be calculated by
- *M* for external given load, m_1 for unit vertical load at ‘C’ and m_2 for unit moment at ‘C’ for various portion of cantilever beam and tabulated below.

Portion	*CB*	*BA*
Origin	*C*	*B*
Limit	0 – 2	0 – 2
M	$-20x$	$-[\,20\,(2+x)+20x\,]$
m_1	$-x$	$-(x+2)$
m_2	–1	–1
I	I_0	$2I_0$

Vertical deflection at 'C' = $\Delta = \int_0^L \frac{Mm_1}{EI} dx$

$$= \int_0^2 \frac{(-20x)(-x)}{EI_o} dx + \int_0^2 \frac{[20(2+x)+20x](x+2)}{E2I_o} dx$$

$$= \int_0^2 \frac{20x^2}{EI_o} dx + \int_0^2 \frac{(40x+40)(x+2)}{2EI_o} dx$$

$$= \left[\frac{20}{3}\frac{x^3}{EI_o}\right]_0^2 + \frac{1}{2EI_o}\left[\frac{40x^3}{3} + \frac{120x^2}{2} + 80x\right]_0^2$$

$$= \frac{53.333}{EI_o} + \frac{1}{EI_o}[253.333]$$

$$= \frac{306.67}{EI_o}$$

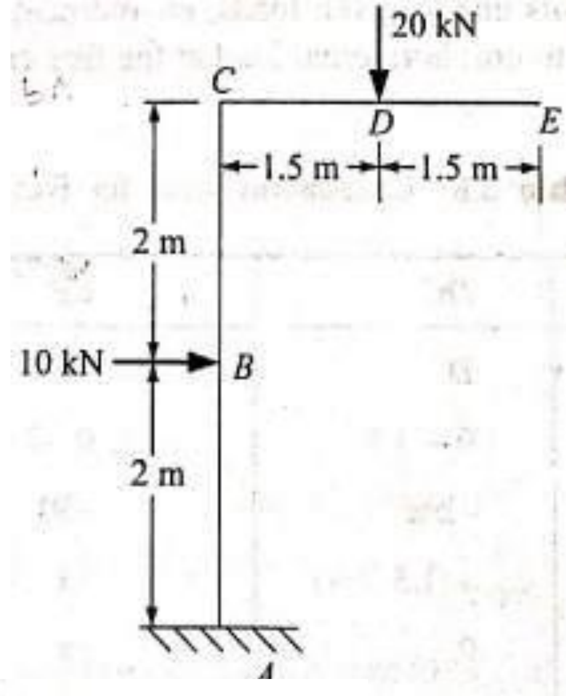
Rotation at 'C' = $\theta_c = \int_0^L \frac{Mm_2}{EI} = \int_0^2 \frac{(20x)}{EI_o} dx + \int_0^2 \frac{(40x+40)1}{E2I_o} dx$

$$= \left[\frac{20}{EI_o} \frac{x^2}{2} \right]_0^2 + \frac{1}{EI_o} \left[\frac{40x^2}{2} + 40x \right]_0^2$$

$$= \frac{40}{EI_o} + \frac{160}{2EI_o}$$

$$= \frac{120}{EI_o}$$

Q3. Determine the vertical and horizontal deflection at the free end of the bent shown in Figure by unit load method. Assume uniform flexural rigidity *EI* throughout.



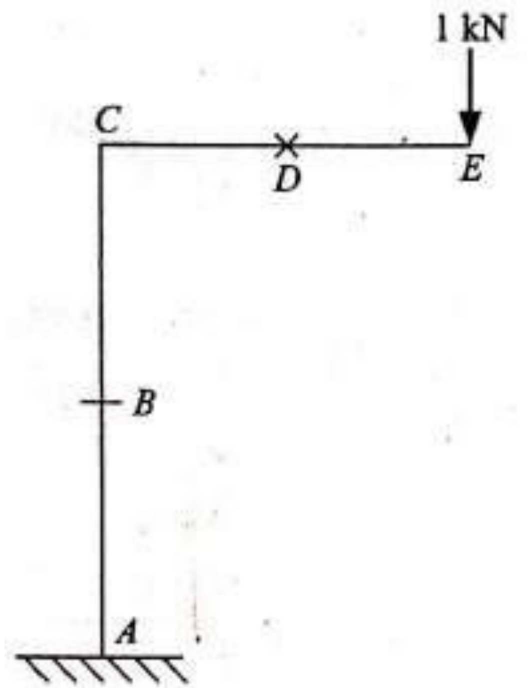


Figure 1: Frame with unit vertical load at ‘*E*’

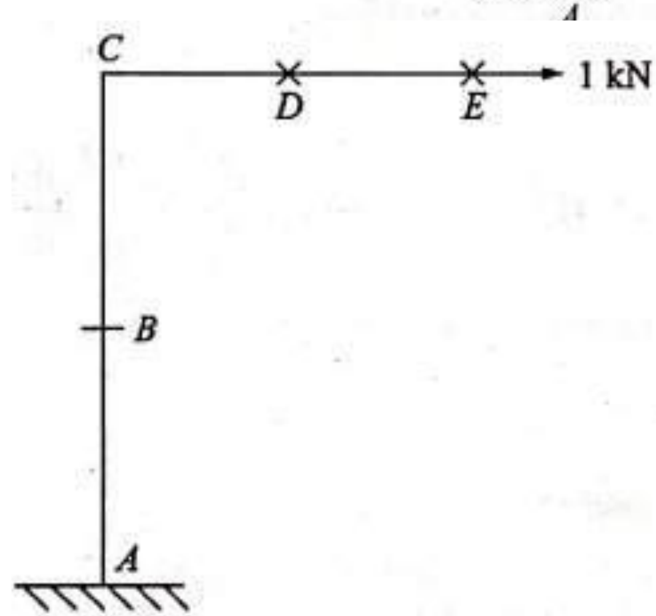


Figure 2: Frame with unit horizontal load at ‘*E*’

- Find out the expressions in Tabular form for moment 'M' due to external loads, m_1 due to the unit vertical load present at the free end (Figure 1) and m_2 due to the unit horizontal load present at the free end (Figure 2) of the bent.

Portion	ED	DC	CB	BA
Origin	E	D	C	B
Limit	0 – 1.5	0 – 1.5	0 –2	0 – 2
M	0	$-20x$	–30	$-30 - 10x$
m_1	x	$-(1.5 + x)$	–3	–3
m_2	0	0	$-x$	$-(x + 2)$
Flexural Rigidity	EI	EI	EI	EI

Note: Moment carrying tension on dotted side is taken as positive

Vertical deflection at 'E' = Δ_{EV}

$$EI\Delta_{EV} = \int Mm_1 \, dx$$

$$= 0 + \int_0^{1.5} 20x(1.5+x)dx + \int_0^{2} 90dx + \int_0^{2} (90+30x)dx$$

$$= \int_0^{1.5} \left(30x+20x^2\right)dx + \int_0^{2} 90dx + \int_0^{2} (90+30x)dx$$

$$= \left[\frac{30x^2}{2} + \frac{20x^3}{3}\right]_0^{1.5} + [90x]_0^2 + \left[90x + \frac{30x^2}{2}\right]_0^2$$

$$= 56.25 + 180 + 240$$

$$= 476.25$$

$$\Delta_{EV} = \frac{476.25}{EI}$$

Horizontal Deflection at '*E*' = Δ_{EH}

$$EI\Delta_{EH} = \int Mm_2\, dx$$

$$= 0+0+\int_0^2 30x\,dx + \int_0^2 (30+10x)(x+2)\,dx$$

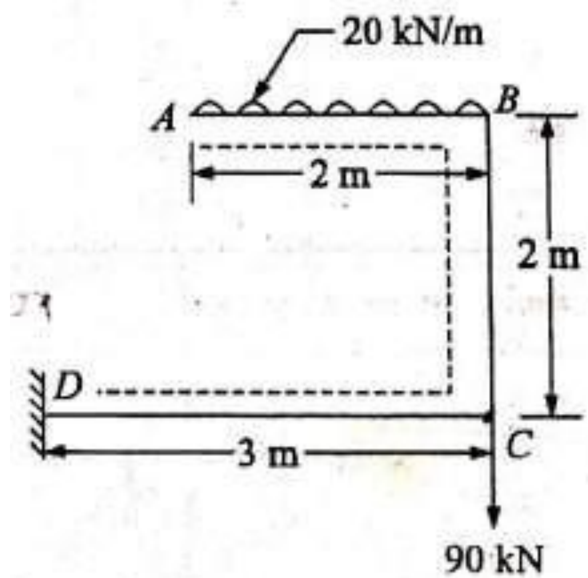
$$= \left[15x^2\right]_0^2 + \int_0^2 \left(10x^2+50x+60\right)dx$$

$$= 60+\left[\frac{10x^3}{3}+50\times\frac{x^2}{2}+60x\right]_0^2$$

$$= 306.67$$

$$\Delta_{EH} = \frac{306.67}{EI}$$

Q4. Determine the vertical deflections at A and C in the frame shown in Figure by unit load method. Take E = 200 GPa, I = 150 × 10^4 mm^4.



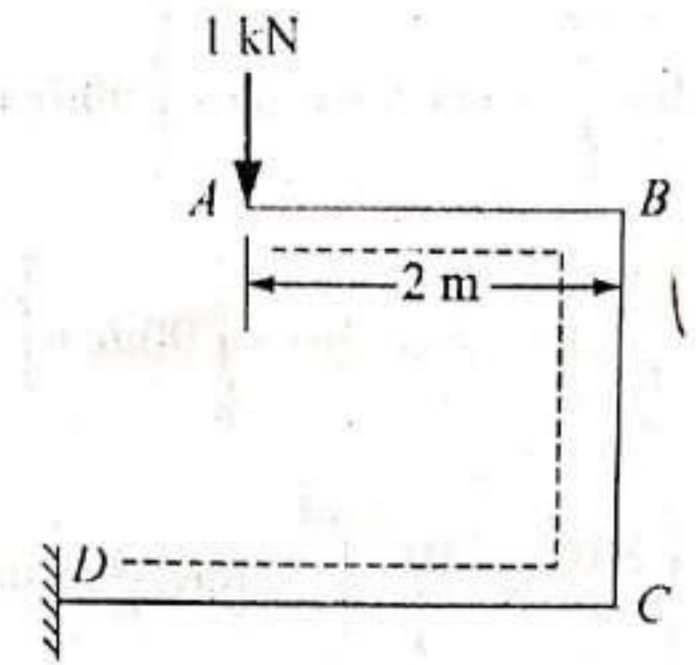


Figure 1: Frame with unit vertical lod at '*A*'

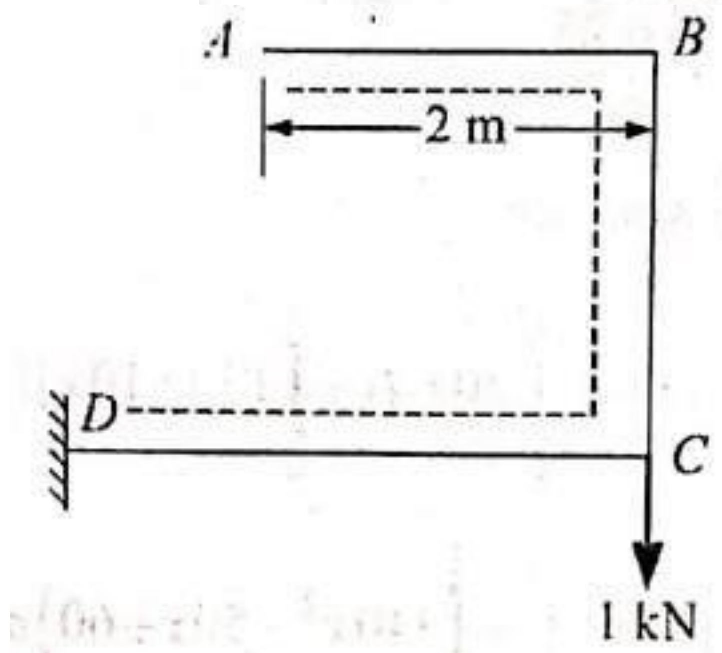


Figure 2: Frame with unit vertical load at '*C*'

- The bending moment expressions for 'M' due to given load, m_1 due to unit vertical load at A and m_2 due to unit vertical load at C are Tabulated below.

Portion	*AB*	*BC*	*CD*
Origin	*A*	*B*	*C*
Limit	0 – 2	0 – 2	0 –3
M	$10x^2$	40	$40 - 130x$
m_1	x	2	$2 - x$
m_2	0	0	$-x$
Flexural Rigidity	*EI*	*EI*	*EI*

Vertical deflection at A = Δ_A

$$EI\Delta_A = \int_0^2 10x^2 . x dx + \int_0^2 80 dx + \int_0^3 (40 - 130x)(2 - x) dx$$

$$= \left[\frac{10x^4}{4}\right]_0^2 + [80x]_0^2 + \int_0^3 \left(80 - 300x + 130x^2\right) dx$$

$$= \frac{10(2^4)}{4} + 80(2) + \left[80x - 300\frac{x^2}{2} + \frac{130x^3}{3}\right]_0^3$$

$$= 260$$

$$E = 240 \text{ GPa} = 240 \times 10^9 \text{ N/m}^2$$

$$I = 150 \times 10^4 \text{ mm}^4 = 150 \times 10^4 \times 10^{-12} \text{ m}^4$$

$$= 150 \times 10^{-8} \text{ m}^4$$

$$\therefore \Delta = \frac{260}{240 \times 10^9 \times 150 \times 10^{-8}} = 7.222 \times 10^{-4} \text{ m}$$

$$= 0.722 \text{ mm}$$

Vertical Deflection at C = Δ_c

$$EI\Delta_c = \int Mm_2\,dx$$

$$= 0+0+\int_0^3 (40-130x)(-x)\,dx$$

$$= \int_0^3 \left(-40x+130x^2\right)dx$$

$$= \left[-20x^2+130\frac{x^3}{3}\right]_0^3$$

$$= 990$$

$$\Delta_C = \frac{990}{240\times10^9\times150\times10^{-8}} = 2.75\times10^{-3}\ \text{m}$$

$$= 2.75\ \text{mm}$$

Thanks