Castigliano's Theorem

Castigliano's First theorem

- The first theorem of Castigliano states that the partial derivative of the total strain energy in any structure with respect to applied force or moment gives the displacement or rotation respectively at the point of application of the force or moment in the direction of the applied force or moment.

$$\frac{dU}{dP_1} = \Delta_1 \qquad \frac{dU}{dM_2} = \theta_2$$

Castigliano's Second Theorem

- The second theorem of castigliano states that the work done by external forces in a structure will be minimum.

- The Theorem is very much useful in analysis of statically indeterminate structures.

Let W = Work done by external forces on a structure

U = Strain energy stored in the structure

W_1 = Work done by reactive forces

Strain Energy = $U = W + W_1$

$$W = U - W_1$$

By Castigliano's 2^{nd} theorem 'W' should be minimum.

Thus the partial derivative of the work done with respect to external forces will be zero.

- In case the supports are unyielding, the work done by reactive forces will be zero.
- Strain energy stored is equal to the work done by external forces will be minimum.
- Thus the partial derivative of strain energy with respect to redundant reaction will be zero.
- Castigliano's First theorem helps in determining deflection of a structure and the Second theorem helps in determining redundant reaction components.

Deflection by Castigliano's Method

Deflection by Castigliano's Method:

Castigliano's theorem may be represented by

$$\frac{dU}{dP_i} = \Delta_i, \quad \frac{dU}{dM_j} = \theta_j \qquad U = \int_0^L \frac{M^2}{2EI} dx$$

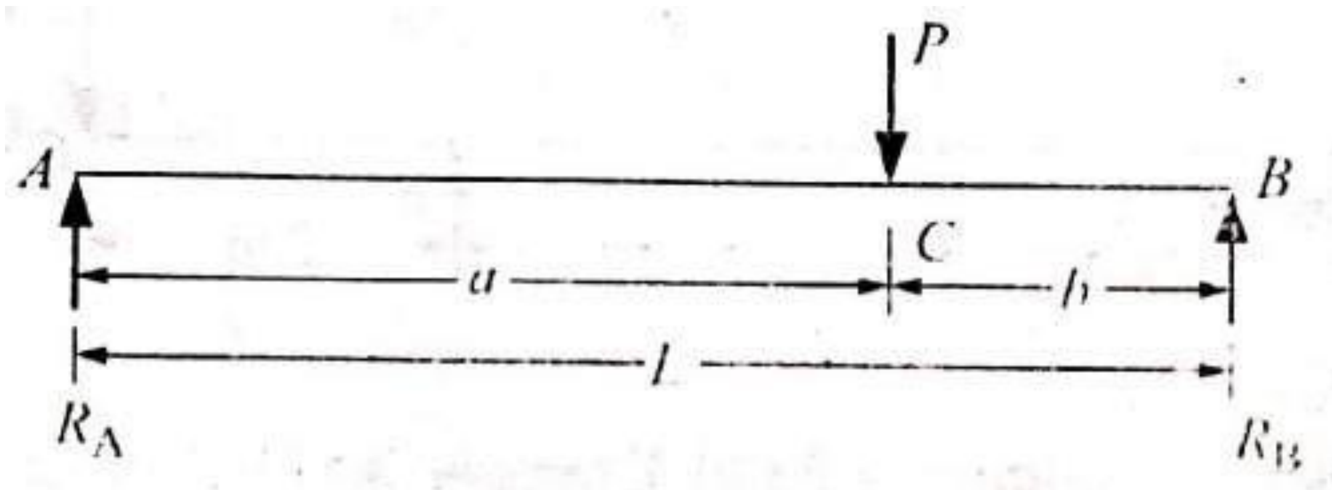
where U = total strain energy

P_i, M_j – loads

Δ_i, θ_j – deflections.

- If a load is acting at a point and is in the desired direction, the general expression for bending moment to cover the entire structure is to be find out.
- The strain energy for the entire structure is differentiated with respect to load (P = Load or M = Moment) to get the desired deflection.
- If the load is not acting, a dummy load (P or M) is applied and then the bending moment expressions is to be find out.
- If dummy load is used, First differentiate w.r.t the dummy load, then substitute dummy load as zero and then integrate w.r.t 'x' .

Q1. A simply supported beam of span '*L*', carries a concentrated load '*P*' at a distance '*a*' from the left hand side as shown in Figure. Using Castigliano's theorem determine the deflection under the load. Assume uniform flexural rigidity.



First determine the reaction by taking moment from any one support,

- Reaction at A, $R_A = \dfrac{Pb}{L}$
- Reaction at B, $R_B = \dfrac{Pa}{L}$
- Find out the expression for moment in a Tabular form for portion BC and then AC.

Portion	AC	CB
Origin	A	B
Limit	0–a	0–b
M	$\frac{Pb}{L}x$	$\frac{Pa}{L}x$
Flexural Rigidity	EI	EI

The strain energy of the Beam = $U = \int_0^L \frac{M^2}{2EI}dx$

$$U = \int_0^a \left(\frac{Pb}{L}x\right)^2 \times \frac{1}{2EI}dx + \int_0^b \left(\frac{Pb}{L}x\right)^2 \times \frac{1}{2EI}dx$$

$$= \left[\frac{P^2b^2}{L^2} \times \frac{1}{6EI} x^3\right]_0^a + \left[\frac{P^2a^2}{L^2} \times \frac{1}{6EI} x^3\right]_0^b$$

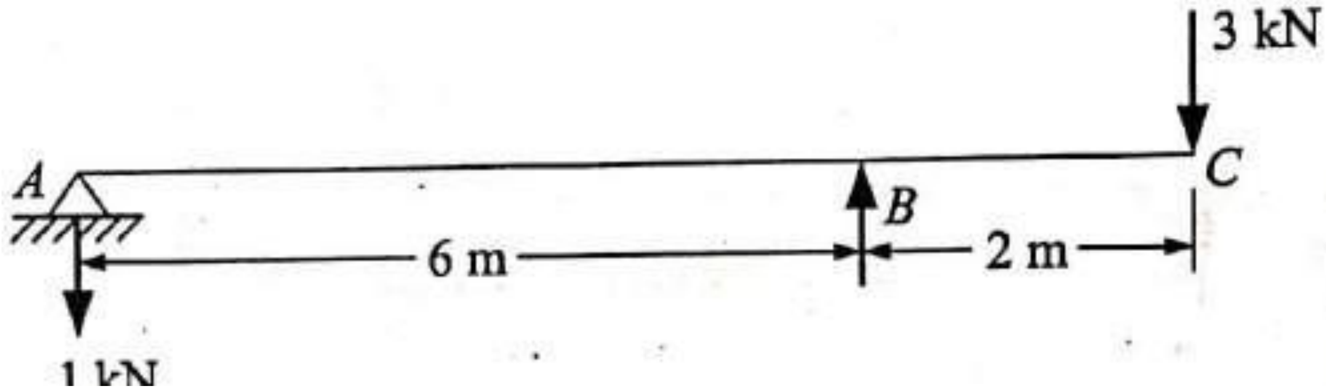
$$= \frac{P^2b^2a^3}{6EIL^2} + \frac{P^2a^2b^3}{6EIL^2}$$

$$= \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

$$= \frac{P^2a^2b^2}{6EIL}, \text{ Since, } a + b = L$$

$$\Delta_C = \frac{\delta U}{\delta P} = \frac{Pa^2b^2}{3EIL}$$

Q2. Determine the vertical deflection at the free end and rotation at ‘*A*’ in the over hanging beam shown in Figure. Use Castigliano’s theorem. Assume uniform flexural rigidity.



Deflection at ‘C’ = Δ_c

- Taking force P = 3 kN and moment about *A*,

$$R_B \times 6 = P \times 8$$

$$R_B = \frac{4}{3}P \uparrow$$

$$R_A = \frac{P}{3} \downarrow$$

Bending moment expression for over hanging beam for portion AB and BC is noted in the Tabular form.

Portion	*AB*	*BC*
Origin	*A*	*C*
Limit	0–6	0–2
M	$\frac{-P}{3}x$	$-Px$
Flexural Rigidity	*EI*	*EI*

$$U = \int \frac{M^2}{2EI} dx$$

$$= \int_0^6 \frac{P^2 x^2}{9} \times \frac{1}{2EI} dx + \int_0^2 \frac{P^2 x^2}{2EI} dx$$

$$= \frac{P^2}{18EI} \left[\frac{x^3}{3} \right]_0^6 + \left[\frac{P^2 x^3}{6EI} \right]_0^2$$

$$= \frac{4P^2}{EI} + \frac{4}{3} \times \frac{P^2}{EI}$$

$$= \frac{5.333P^2}{EI}$$

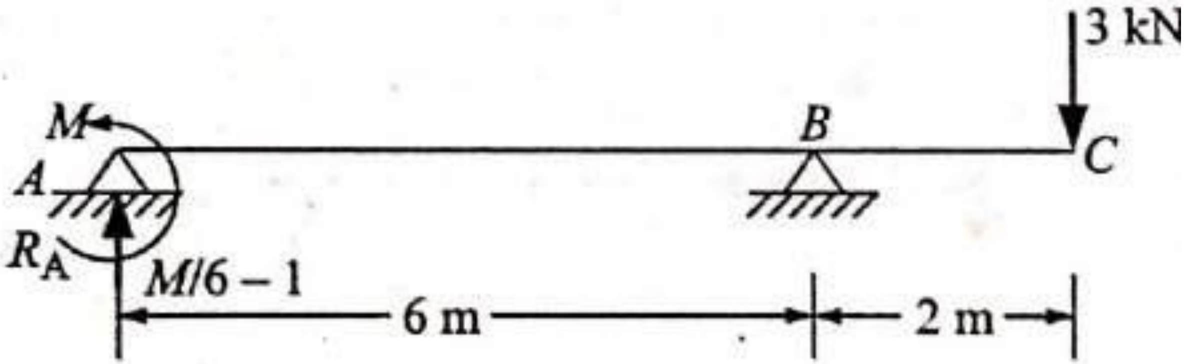
$$\Delta_C = \frac{dU}{dP} = \frac{10.667P}{EI}$$

Substituting $P = 3$ kN, we get

$$\Delta_C = \frac{32}{EI}$$

Rotation at A = θ_A

- Apply dummy moment, 'M' at A as shown in Figure



$$\Sigma M_B = 0, \text{ gives}$$

$$R_A = \frac{M-6}{6} = \frac{M}{6} - 1$$

Portion	AB	BC
Origin	A	C
Limit	0 – 6	0 – 2
M	$\left(\frac{M}{6}-1\right)x - M$	$-3x$

$$U = \int_0^6 \left[\left(\frac{M}{6} - 1\right)x - M\right]_0^2 \frac{1}{2EI}\,dx + \int_0^2 \frac{(-3x)^2}{2EI}\,dx$$

$$\frac{dU}{dM} = \int_0^6 2\left[\left(\frac{M}{6} - 1\right)x - M\right]\left(\frac{x}{6} - 1\right)\frac{dx}{2EI} + 0$$

Since, '*M*' is a dummy moment, its value is substituted as zero, and then integrated

$$\frac{dU}{dM} = \theta_A = \frac{1}{EI}\int_0^6 (-x)\left(\frac{x}{6} - 1\right)dx$$

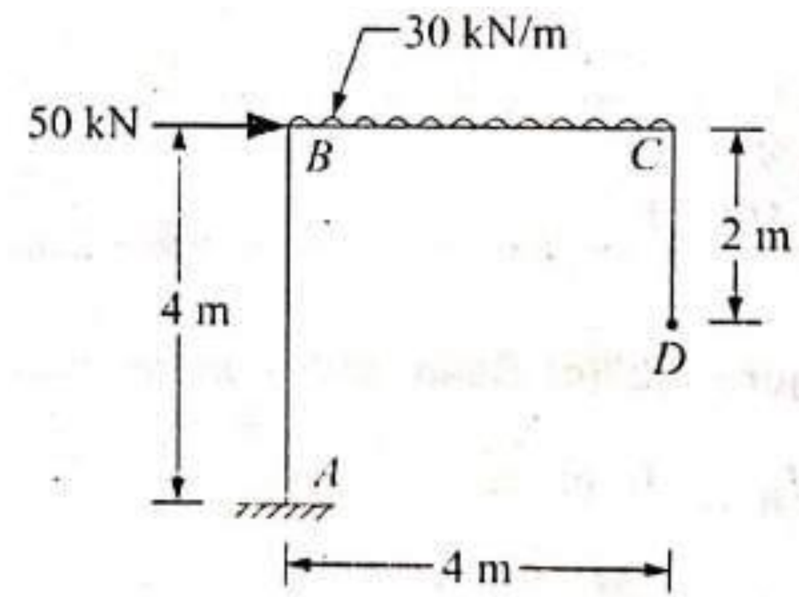
$$= \frac{1}{EI}\int_0^6 \left(-\frac{x^2}{6} + x\right)dx$$

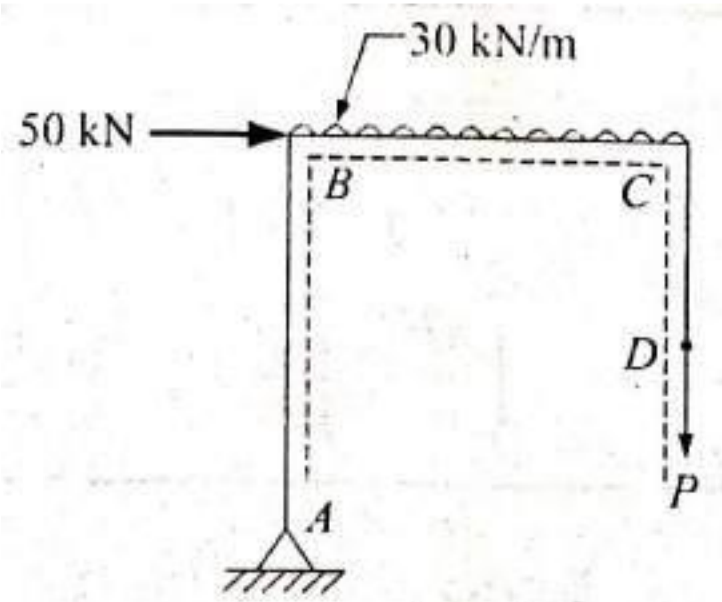
$$= \frac{1}{EI}\left(-\frac{x^3}{18} + \frac{x^2}{2}\right)_0^6 dx$$

$$= \frac{6}{EI}$$

Note: First differentiate w.r.t the dummy load, then substitute dummy load as zero and then integrate w.r.t '*x*'.

Q2. Determine the vertical and horizontal deflection at the free end '*D*' in the frame shown in Figure. Use Castigliano's theorem. Take $EI = 12\times 10^{13}$ Nmm².





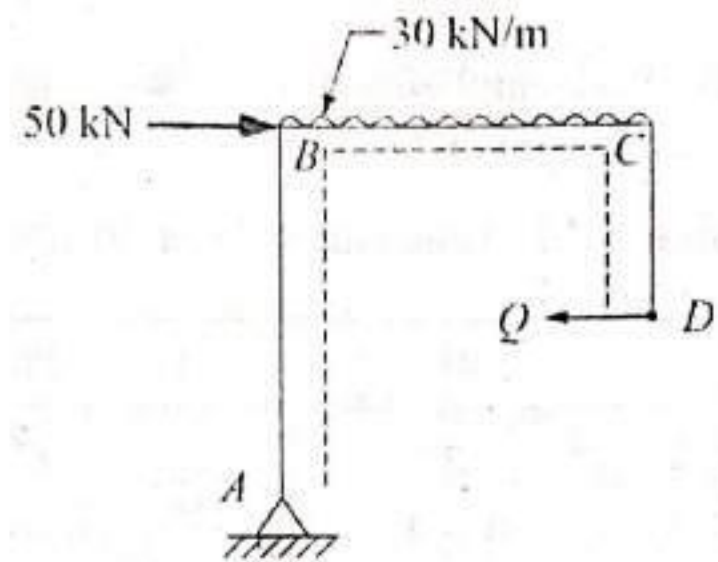


Figure 2: Frame with dummy horizontal load '*Q*' at '*D*'

Vertical Deflection:

- Since, there is no load at '*D*' in vertical direction, a dummy load '*P*' is applied at '*D*' in vertical direction in addition to given loads as shown in Figure 1. The moment expressions are presented in a tabular form.

Portion	*AB*	*BC*	CD
Origin	*B*	*C*	*D*
Limit	0 – 4	0 – 4	0 – 2
M	$-(4P + 240 + 50x)$	$-(Px + 15x^2)$	0
Flexural Rigidity	*EI*	*EI*	*EI*

$$\text{Strain energy } U = \int \frac{M^2}{2EI} dx$$

$$= \int_0^4 \frac{(4P+240+50x)^2}{2EI} dx + \int_0^4 \frac{\left(Px+15x^2\right)^2}{2EI} dx + 0$$

$$\Delta = \frac{\delta U}{\delta P} = \int_0^4 2\frac{(4P+240+50x)}{2EI}(4)\, dx + \int_0^4 2\frac{\left(Px+15x^2\right)x}{2EI} dx$$

Since, P is dummy load, substitute $P = 0$

$$\Delta_D = \int_0^4 \frac{4(240+50x)}{EI} dx + \int_0^4 \frac{15x^3}{EI} dx$$

$$= \frac{4}{EI}\left[240x + 25x^2\right]_0^4 + \frac{15}{EI}\left(\frac{x^4}{4}\right)_0^4 = \frac{6400}{EI}$$

Now,

$$EI = 12 \times 10^{13} \text{ Nmm}^2$$

$$= 12 \times 10^4 \text{ kNm}^2$$

$$\therefore \quad \Delta_{DV} = \frac{6400}{12 \times 10^4} = 0.533 \text{ m}$$

$$= 53.33 \text{ mm}$$

Horizontal Deflection:

- Since, there is no load at '*D*' in horizontal direction, a dummy load '*Q*' is applied at '*D*' in horizontal direction in addition to given loads as shown in Figure 2. The moment expressions are presented in a tabular form.

Portion	AB	BC	CD
Origin	B	C	D
Limit	0 – 4	0 – 4	0 – 2
M	$-[Q(2-x)+240+50x)]$	$-(2Q+15x^2)$	Qx
Flexural Rigidity	EI	EI	EI

$$U=\int_0^4 \frac{\left[Q(2-x)+240+50x\right]^2}{2EI}dx+\int_0^4 \frac{\left[\left(2Q+15x^2\right)\right]^2}{2EI}dx+\int_0^2 \frac{Q^2x^2}{2EI}dx$$

$$\Delta_{DH}=\frac{dU}{dQ}=\int_0^4 \frac{2\left[Q(2-x)+240+50x\right](2-x)}{2EI}dx+\int_0^4 \frac{2\left[2Q+15x^2\right]2}{2EI}dx+\int_0^2 \frac{2Qx^2}{2EI}dx$$

Substituting $Q = 0$

$$\Delta_{DH} = \int_0^4 \frac{(240+50x)(2-x)}{EI}dx + \int_0^4 \frac{30x^2}{EI}dx + 0$$

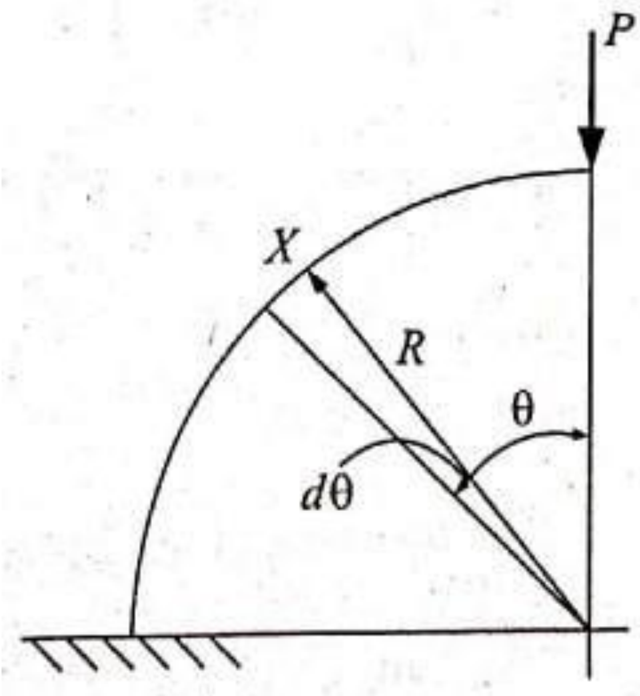
$$= \int_0^4 \frac{\left(480-140x-50x^2\right)}{EI}dx + \int_0^4 \frac{30x^2}{EI}dx$$

$$= \frac{1}{EI}\left[480x - 70x^2 - \frac{50x^3}{3}\right]_0^4 + \frac{1}{EI}\left[10x^3\right]_0^4$$

$$= \frac{373.33}{EI} = \frac{373.33}{12\times10^4} = 0.0031 \text{ m}$$

$$= 3.1 \text{ mm}$$

Q2. A cantilever beam is in the form of a quarter of a circle in the vertical plane and is subjected to a vertical load 'P' at its free end as shown in Figure. Find the vertical and horizontal deflections at the free end. Use Castigliano's theorem. Assume uniform flexural rigidity.



Vertical Deflection of free end:

- Consider the section at 'x' as shown in Figure 1. The Bending moment at the section 'x' is

$$M = PR \sin \theta$$

Strain energy in the elemental length '$R\,d\theta$' is

$$= \left(\frac{M^2}{2EI}\right) R\,d\theta$$

$$= \frac{P^2 R^2 \sin^2\theta}{2EI} R\,d\theta$$

$$= \frac{P^2 R^3}{2EI} \times \frac{1-\cos 2\theta}{2} d\theta$$

$$U = \int_0^{\pi/2} \frac{P^2 R^3}{2EI} \times \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{P^2 R^3}{4EI}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi P^2 R^3}{8EI}$$

$$\Delta_V = \frac{\delta U}{dP} = \frac{\pi P R^3}{4EI}$$

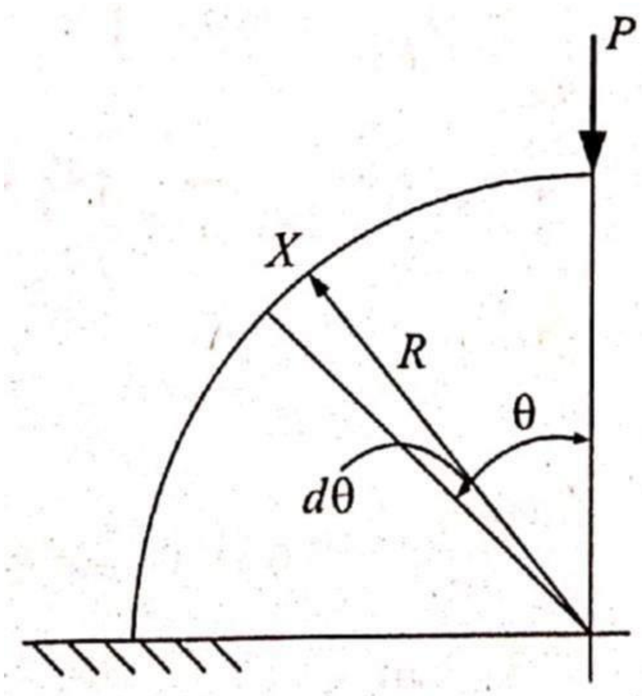


Figure 1: Cantilever curved beam

Horizontal Deflection:

Since, there is no horizontal force at the free end, apply a dummy horizontal force '*Q*', as shown in Figure 2.

The bending moment at section '*x*' is

$$M = PR\sin\theta + QR(1-\cos\theta)$$

Strain Energy (*U*) =

$$U = \int_0^{\pi/2} \frac{\left[PR\sin\theta + QR(1-\cos\theta)\right]^2}{2EI} R\, d\theta$$

Horizontal Displacement = Δ_H

$$\Delta_H = \frac{\delta U}{\delta Q} = \int_0^{\pi/2} \frac{\left[PR\sin\theta + QR(1-\cos\theta)\right]}{EI}\left[R(1-\cos\theta)\right] R\, d\theta$$

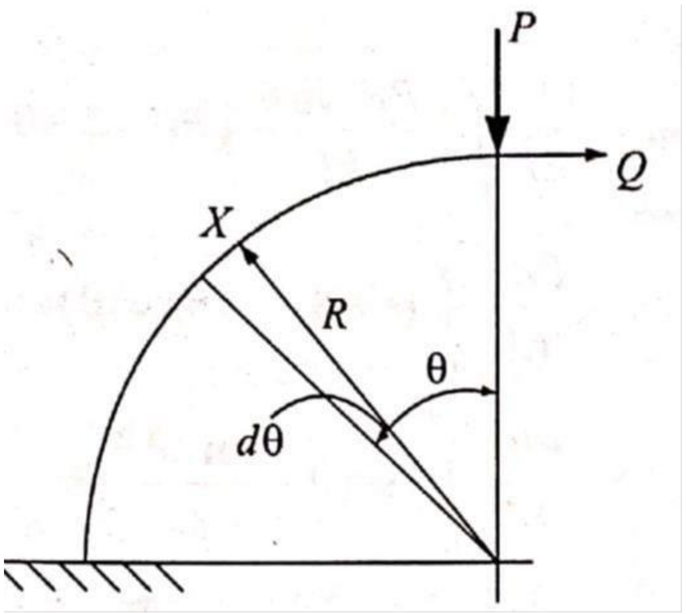


Figure 2: Cantilever curved beam with dummy load '*Q*' at the free end

Substituting, Q = 0, in above equation

$$\Delta_H = \frac{1}{EI}\int_0^{\pi/2}\left(\frac{PR\sin\theta}{EI}\right)[R(1-\cos\theta)]R\,d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}(\sin\theta-\sin\theta\cos\theta)\,d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}\left(\sin\theta-\frac{\sin 2\theta}{2}\right)d\theta$$

$$= \frac{PR^3}{EI}\left(\cos\theta-\frac{\cos 2\theta}{4}\right)_0^{\pi/2}$$

$$= \frac{PR^3}{EI}\left(0+\frac{1}{4}-1+\frac{1}{4}\right)$$

$$= -\frac{PR^3}{2EI}$$

i.e., $\Delta_H = \frac{PR^3}{2EI}$, towards support