# Deflection by Strain Energy Method

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- This Method is also called 'Real Work Method'.
- Since, work done by the actual loads are considered.
- From the law of conservation of energy,

  **Strain Energy (U) = Real work done by loads**

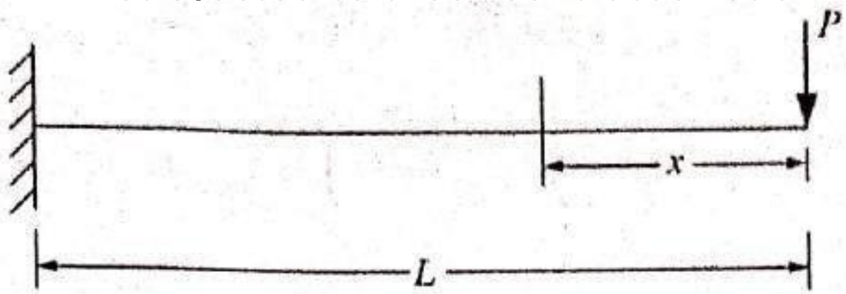
$$U = \sum_{0}^{n} \frac{1}{2} P\Delta$$

- This equation can be used to find out the deflection in beams and frames subjected to bending stresses.

**Strain energy method can be used for finding deflection under the following situations**:

- The structure is subjected to a concentrated load.
- Deflection required is at the loaded point and is in the direction of load.

Q1. Using strain energy method determine the deflection of the free end of a cantilever of length ‘$L$’ subjected to a concentrated load ‘$P$’ at the free end.




**Solution** The bending moment at a distance $x$ from the free end is,

$$M = Px$$

$$\therefore \quad \text{Strain Energy (S.E.)} = \int_0^L \frac{M^2}{2EI} dx$$

$$= \int_0^L \frac{P^2 x^2}{2EI} dx$$

$$= \frac{P^2}{2EI} \left[ \frac{x^3}{3} \right]_0^L$$

$$= \frac{P^2 L^3}{6EI}$$

Work done by the load $= \frac{1}{2}P\Delta$, where $\Delta$ is the deflection at the free end.

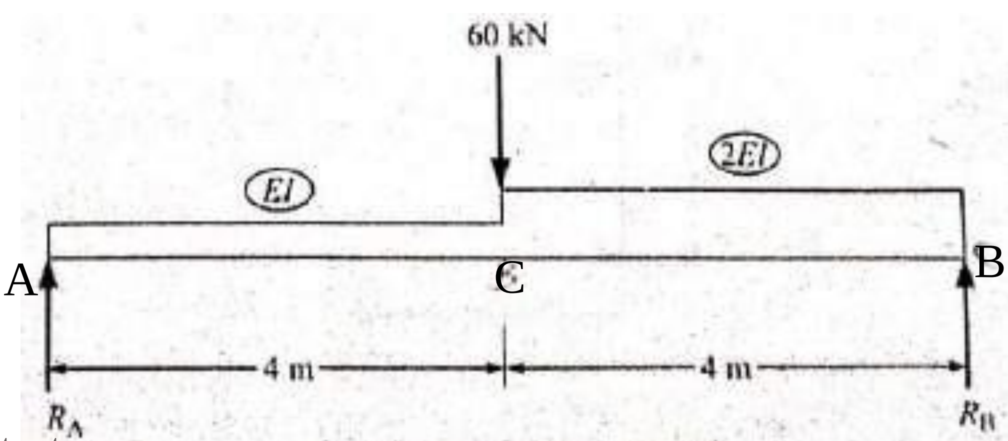
Therefore, from conservation of energy,

S.E. = Work done by external loads

$$\frac{P^2L^3}{6EI} = \frac{1}{2}P\Delta$$

$$\Delta = \frac{PL^3}{3EI}$$

Q2. Using strain energy method determine the deflection under 60 kN load in the beam shown in Figure.



**Solution** Reaction $R_A = R_B = 30$ kN

Therefore, bending moment at any distance $x$ from $A$ or at a distance $x$ from $B$

$$= 30x \text{ kN}$$

$$\therefore \quad \text{S.E.} = \int_0^4 \frac{(30x)^2}{2EI}dx + \int_0^4 \frac{(30x)^2}{2\times 2EI}dx$$

$$U = \frac{3}{4}\times\frac{900}{EI}\int_0^4 x^2\,dx$$

$$U = \frac{3}{4}\times\frac{900}{EI}\left[\frac{x^3}{3}\right]_0^4 = \frac{3}{4}\times\frac{900}{EI}\times\frac{4^3}{3}$$

$$U = \frac{14400}{EI}$$

Work done by the load:

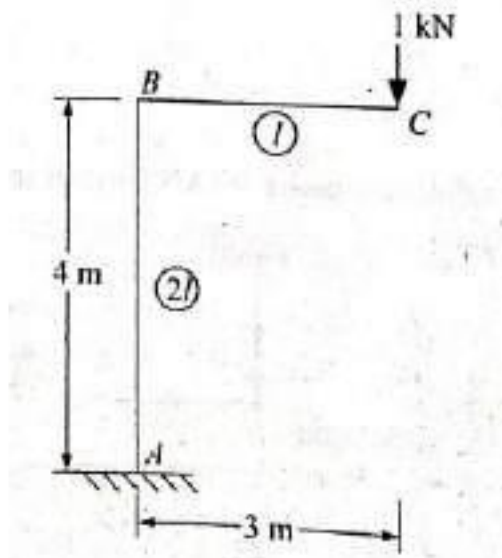
$$W_E = \frac{1}{2}\times P\Delta = \frac{1}{2}\times 60\times\Delta$$

Equating strain energy of the beam to the work done by load; we get,

$$\frac{14400}{EI} = \frac{1}{2}\times 60\times\Delta$$

$$\Delta = \frac{480}{EI}$$

Q3. Using strain energy method determine the vertical deflection of point '$C$' in the frame shown in Figure. $E = 200$ kN/mm$^2$ and $I = 30 \times 10^6$ mm$^4$.




- The details of bending moment expressions for various portion of the structure is calculated individually for member BC than for member AB, and given data in Tabular form:

| *Portion* | *Origin* | *Limit* | *Expression* |
|---|---|---|---|
| *BC* | *C* | 0 – 3 | $1\,x = x$ |
| *AB* | *B* | 0 – 4 | 3 |

$$\text{S.E} = \int_0^3 \frac{(x)^2}{2EI}dx + \int_0^4 \frac{(3)^2}{2E\times 2I}dx$$

$$= \frac{1}{2EI}\left[\frac{x^3}{3}\right]_0^3 + \frac{1}{4EI}[9x]_0^4$$

$$= \frac{1}{6EI}\times 3^3 + \frac{1}{4EI}\times 9\times 4$$

$$= \frac{13.5}{EI}$$

Note: As the bending moment is given in kN and metres, $EI$ should be used as kNm$^2$.
i.e. 1 kNmm$^2$ = 1× 10$^{-6}$ kNm$^2$

$$\text{Work done} = \frac{1}{2}\times 1\times\Delta = \frac{\Delta}{2}$$

Equating work done to strain energy, we get

$$\frac{\Delta}{2} = \frac{13.5}{EI}$$

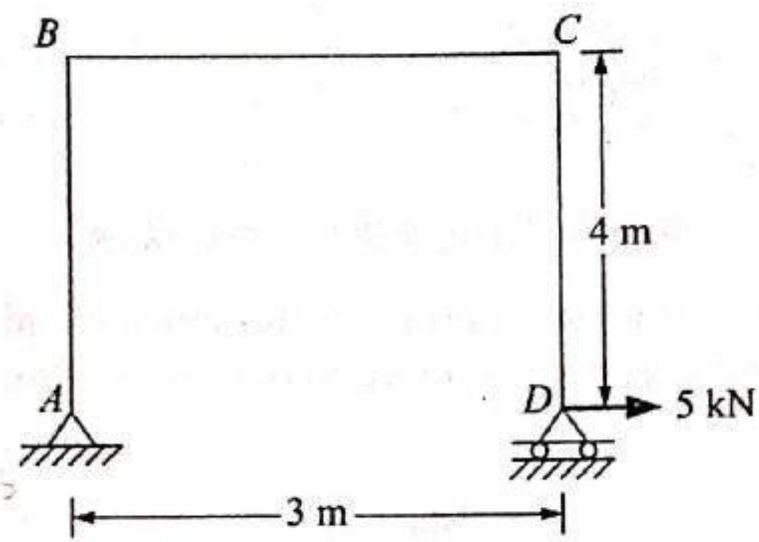
$$\Delta = \frac{27}{EI}$$

$$\text{EI} = 200\times 30\times 10^6\times 10^{-6} = 6000\ \text{kNm}^2$$

$$\Delta = \frac{27}{6000}\ \text{m}$$

$$= 0.045\ \text{m} = 4.5\ \text{mm}$$

Q4. Using strain energy method determine the horizontal deflection of the roller end ‘*D*’ of the portal frame shown in Figure. *EI* = 8000 kNm² throughout.




- The details of bending moment expressions for various portion of the structure is calculated individually for member CD, BC than for member AB, and given data in Tabular form:

| *Portion* | *CD* | *BC* | *AB* |
|---|---|---|---|
| Origin | *D* | *C* | *B* |
| Limit | 0 – 4 | 0 – 3 | 0 – 4 |
| $M_x$ | 5x | 20 | 20 – 5x |

$$S.E = \int_0^4 \frac{(5x)^2}{2EI}dx + \int_0^3 \frac{(20)^2}{2EI}dx + \int_0^4 \frac{(20-5x)^2}{2EI}dx$$

$$= \frac{1}{2EI}\left[\frac{25x^3}{3}\right]_0^4 + \frac{1}{2EI}[400x]_0^3 + \frac{1}{2EI}\left[400x - 200\frac{x^2}{2} + \frac{25x^3}{3}\right]_0^4$$

$$= \frac{266.67}{EI} + \frac{600}{EI} + \frac{1}{2EI}\left[1600 - 1600 + \frac{25 \times 64}{3}\right]$$

$$= \frac{1133.33}{EI}$$

Work done $= \frac{1}{2} \times P \times \Delta = \frac{1}{2} \times 5\Delta = 2.5\Delta$

Equating S.E. to work done, we get, $2.5\Delta = \frac{1133.33}{EI}$

$$\Delta = \frac{453.33}{EI} = \frac{453.33}{8000} = 0.0567\,\text{m}$$

$$= 56.7\,\text{mm}$$