Subject Code & Name: 19AST203 Aircraft Structural Mechanics

DAY: **01** TOPIC: **Strain Energy in axial**

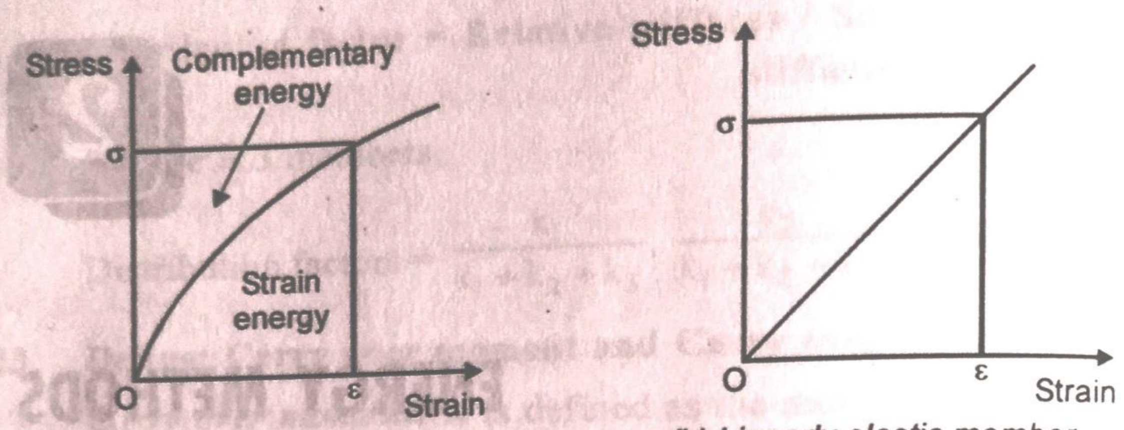
2

ENERGY METHODS

Strain Energy in axial, bending, torsion and shear loadings. Castigliano's theorems and their applications. Energy theorems – dummy load and unit load methods – Energy methods applied to statically determinate and indeterminate beams, frames, rings and trusses.

2.1 STRAIN ENERGY

When an elastic body is subjected to some external load(s), it undergoes deformation and regain its original position after removal of the applied load. For the loaded time, some amount of energy stored on it, the same is given up or released when the load is removed. This energy is called strain energy (U). The strain energy per unit volume is defined as the strain energy density (U*). The strain energy stored by the body 'within' the elastic limit, when the load is applied externally is called 'Resilience'. The maximum energy stored by a body 'up to' the elastic limit is called 'Proof resilience' and proof resilience per unit volume is called 'Modulus of resilience'. Also the strain energy stored in a body upto the breaking point is called 'Toughness', which is the measure of maximum energy that a component can store before failure. Graphically (Refer Figure 2.1) the area under the stress-strain diagram, upto the maximum stress developed in the component due to the applied load is the strain energy density (U*) and the area above this curve with same stress is the complementry strain energy density (U*)'.



(a) Non-linearly elastic member (b) Linearly elastic member

Figure 2.1: Stress-Strain diagram

2.1.1 Strain energy in axial loading

Let us consider a bar of cross-sectional area A and length ℓ and subjected to the axial load P (Figure 2.2). Suppose this load extends the bar by a small displacement δl and produces a maximum stress σ.

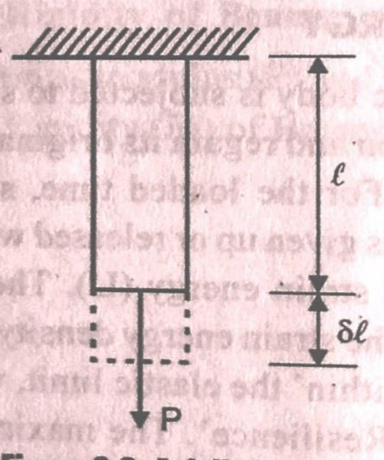


Figure 2.2: Axially loaded bar

The workdone by P and hence the strain energy stored in the material is equal (i.e), U = W.

$$\text{Work done,} \quad W = \text{Average force} \times \text{distance}$$

$$= \frac{P}{2} \times \delta\ell$$

$$= \frac{1}{2} P \, \delta\ell$$

We already discussed above, U = W

Hence, $$U = \frac{1}{2} P\, \delta\ell \qquad \text{... (2.1)}$$

We know that, the deflection, $\delta\ell = \frac{P\ell}{AE}$ (from Hooke's law).

$$\therefore \quad U = \frac{1}{2} P \left(\frac{P\ell}{AE}\right)$$

$$U = \frac{P^2 \ell}{2AE}$$

If we consider a small element in the bar, the above expression is changed to,

$$dU = \frac{P^2}{2AE} dx$$

While integrating this equation, we get

$$\boxed{U_A = \int_0^{\ell} \frac{P^2}{2AE} dx} \qquad \text{... (2.2)}$$

The above equation is the general form of strain energy in axial loaded structures. It also can be rewritten as,

$$\boxed{U_A = \int_0^{\ell} \frac{\sigma^2}{2E} dV} \qquad \text{... (2.3)}$$

Where, $\sigma = \frac{P}{A}$ and $dV = A \times dx$

This equation is also used to solve the pin-jointed frame structures.

2.1.2 Strain energy in bending loads

Let us consider a beam of uniform cross-section with certain end conditions such that the bending moment varies along its length. Consider a small length dx of a beam where the bending moment is M (Figure 2.3).

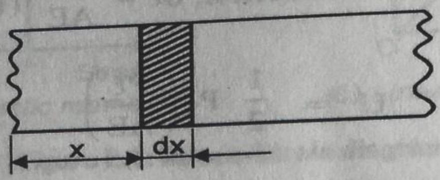


Figure 2.3: Beam with bending moment

From equation 2.3, we know that,

$$U = \int \frac{\sigma^2}{2E} dV$$

If there will be change in dimensions of a cross-sectional area, dV becomes dA × dx and hence triple integration requies to solve the equation. So equation 2.3 can be rewritten as,

$$U = \iiint \frac{\sigma^2}{2E} dA\, dx$$

From the equation of simple bending, $\frac{\sigma}{y} = \frac{M}{I}$.

$$\therefore \quad U = \iiint \frac{\left(\frac{M}{I} y\right)^2}{2E} dA\, dx$$

$$= \iiint \frac{1}{2E}\left(\frac{M}{I}\right)^2 dx\, y^2\, dA$$

$$= \int \frac{1}{2E}\left(\frac{M}{I}\right)^2 dx \iint y^2 \, dA$$

$$= \int \frac{M^2}{2EI^2} dx \, I \quad (\text{where, } I_{xx} = \iint y^2 \, dA)$$

$$\boxed{U_B = \int \frac{M^2}{2EI^2} dx} \qquad \text{... (2.4)}$$

The above equation is the general form of strain energy of the structures subjected to bending moment. This equation is also used to solve the rigid-jointed frame structures.

2.1.3 Strain energy in Pure shear loading

Let us consider a section subjected to shearing force Q acting across two of its opposite forces (Figure 2.4). The workdone by Q in a shear displacement of element 'dx' is given by,

$$U = \frac{1}{2} \times Q \times (\text{Shear displacement of element dx})$$

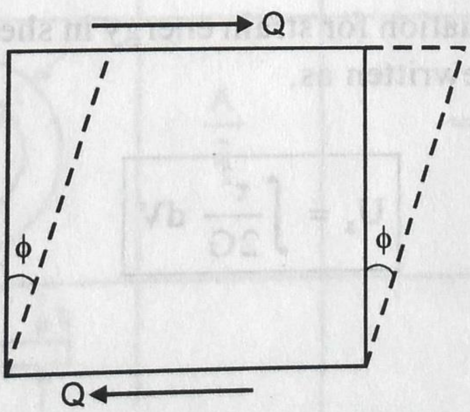


Figure 2.4 Shear loaded structure

$$\text{Shear displacement, dx} = \frac{\tau}{G} x$$