Two-Dimensional Geometric Transformations

Two-Dimensional Geometric Transformations

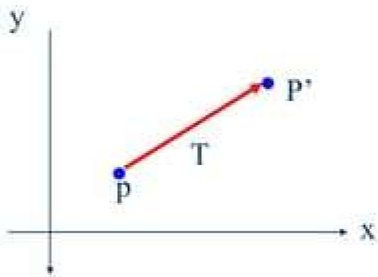
- **Basic Transformations**
  - Translation
  - Rotation
  - Scaling
- **Composite Transformations**
- **Other transformations**
  - Reflection
  - Shear

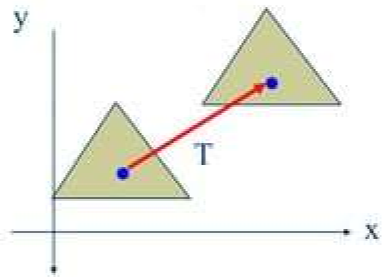
Translation

- **Translation transformation**

$$y' = y + t_y$$

  - **Translation vector or shift vector $T = (t_x, t_y)$**
- **Rigid-body transformation**
  - **Moves objects without deformation**





Rotation

Rotation transformation (anticlockwise)

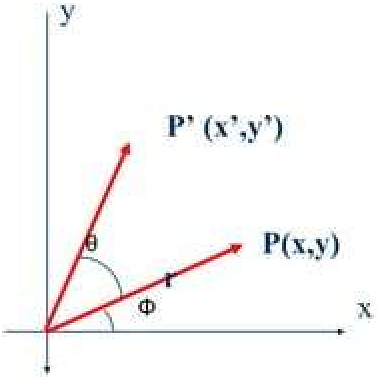
$x' = r\cos(\theta+\Phi) = r\cos\theta\cos\Phi - r\sin\theta\sin\Phi$

$y' = r\sin(\theta+\Phi) = r\cos\theta\sin\Phi + r\sin\theta\cos\Phi$

$x = r\cos\Phi \quad y = r\sin\Phi$

$x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta$



$P' = R \cdot P$

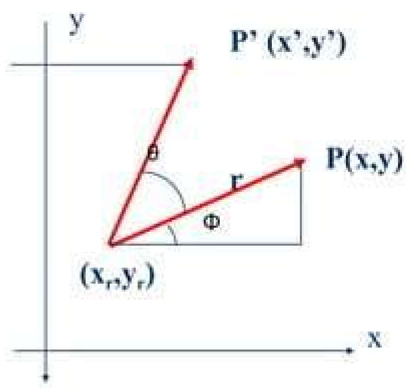
$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation

- **General Pivot point rotation**

$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$

$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$

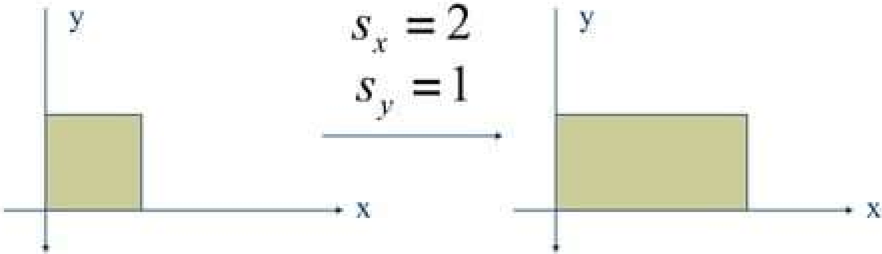


Scaling

- **Scaling transformation**

$$\begin{cases} x' = x \cdot s_x \\ y' = y \cdot s_y \end{cases} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = S \cdot P$$

- **Scaling factors, s_x and s_y**



Scaling

- **Fixed point**

$$\begin{cases} x' = x_f + (x - x_f) \cdot s_x \\ y' = y_f + (y - y_f) \cdot s_y \end{cases}$$

$$\begin{cases} x' = x \cdot s_x + x_f (1 - s_x) \\ y' = y \cdot s_y + y_f (1 - s_y) \end{cases}$$

Matrix Representations and Homogeneous Coordinates

- **Homogeneous Coordinates**

$$(x, y) \rightarrow (x_h, y_h, h)$$

$$x = \frac{x_h}{h}$$

$$y = \frac{y_h}{h}$$

- **Matrix representations**
  - Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Matrix Representations

- **Matrix representations**
  - Scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

  - Rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Composite Transformations

- Translations

$$
\begin{aligned}
P' &= T(t_{x2}, t_{y2}) \cdot \{T(t_{x1}, t_{y1}) \cdot P\} \\
&= \{T(t_{x2}, t_{y2}) \cdot T(t_{x1}, t_{y1})\} \cdot P
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
T(t_{x2}, t_{y2}) \cdot T(t_{x1}, t_{y1}) = T(t_{x1} + t_{x2}, t_{y1} + t_{y2})
$$

Composite Transformations

- **Scaling**

$$\begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S(s_{x2}, s_{y2}) \cdot S(s_{x1}, s_{y1}) = S(s_{x1} * s_{x2}, s_{y1} * s_{y2})$$

Composite Transformations

- **Rotations**

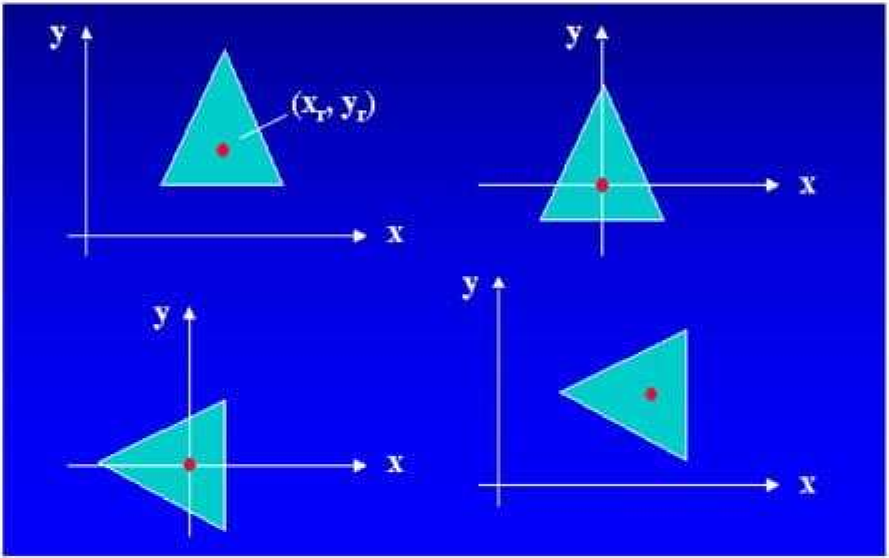
$$P^{'} = R(\theta_2)\cdot\{R(\theta_1)\cdot P\}$$
$$= \{R(\theta_2)\cdot R(\theta_1)\}\cdot P$$

$$\begin{bmatrix}\cos\theta_2 & -\sin\theta_2 & 0\\ \sin\theta_2 & \cos\theta_2 & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\cos\theta_1 & -\sin\theta_1 & 0\\ \sin\theta_1 & \cos\theta_1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$R(\theta_2)\cdot R(\theta_1) = R(\theta_1+\theta_2)$$

General Pivot-Point Rotation

- Rotations about any selected pivot point (x_r, y_r)
  - Translate-rotate-translate



General Pivot-Point Rotation

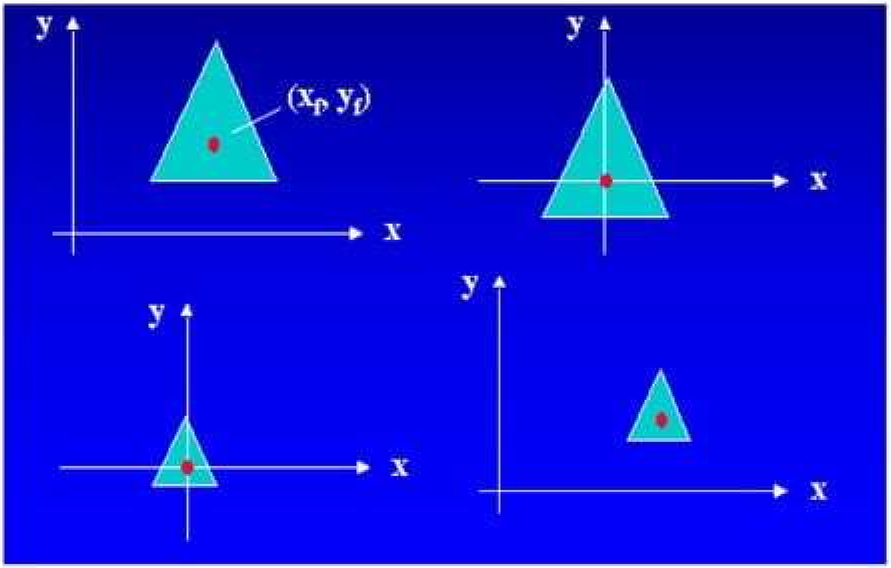
$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(x_r, y_r) \cdot R(\theta) \cdot T(-x_r, -y_r) = R(x_r, y_r, \theta)$$

General Fixed-Point Scaling

Scaling with respect to a selected fixed position (x_f, y_f)



General Fixed-Point Scaling

- Translate-scale-translate

$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(x_f, y_f) \cdot S(s_x, s_y) \cdot T(-x_f, -y_f) = S(x_f, y_f, x_r, y_r)$$

Concatenation Properties

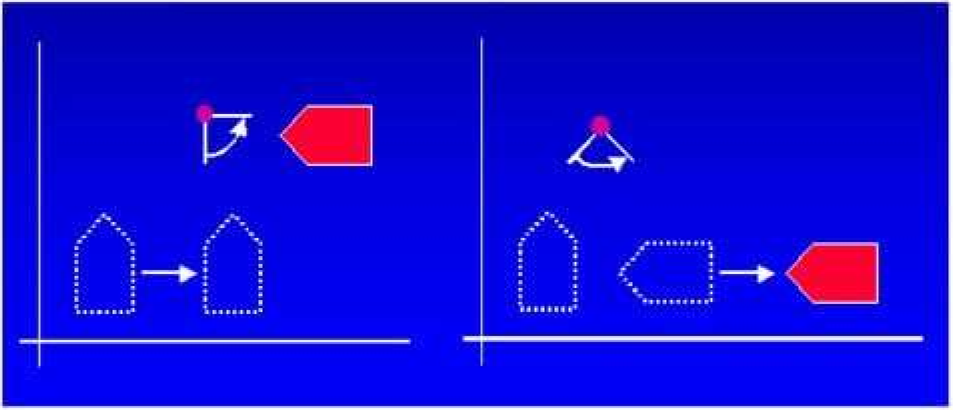
- **Matrix multiplication is associative.**

  $$A \cdot B \cdot C = (A \cdot B) \cdot C = A \cdot (B \cdot C)$$

- **Transformation products may not be commutative**
  - Be careful about the order in which the composite matrix is evaluated.
  - Except for some special cases:
    - Two successive rotations
    - Two successive translations
    - Two successive scalings
    - rotation and uniform scaling

Concatenation Properties

- **Reversing the order**
  - A sequence of transformations is performed may affect the transformed position of an object.

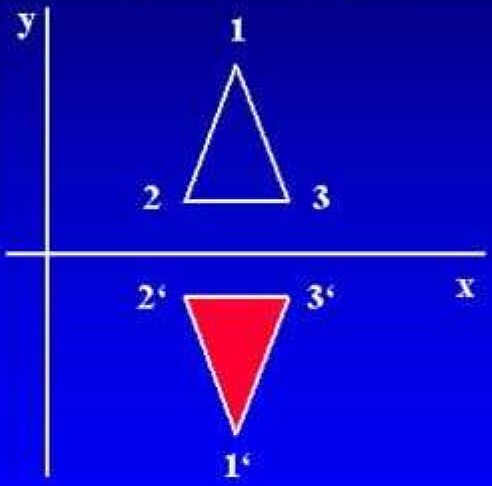


Reflection

- A transformation produces a mirror image of an object.
- Axis of reflection
  - A line in the xy plane
  - A line perpendicular to the xy plane
  - The mirror image is obtained by rotating the object 180' about the reflection axis.
- Rotation path
  - Axis in xy plane: in a plane perpendicular to the xy plane.
  - Axis perpendicular to xy plane: in the xy plane.

Reflection

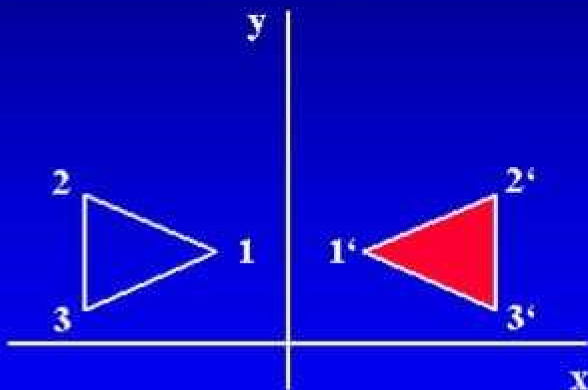
- Reflection about the x axis



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection

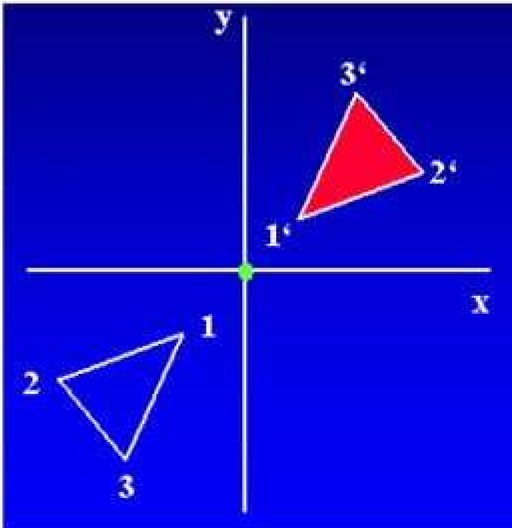
- Reflection about the y axis



$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection

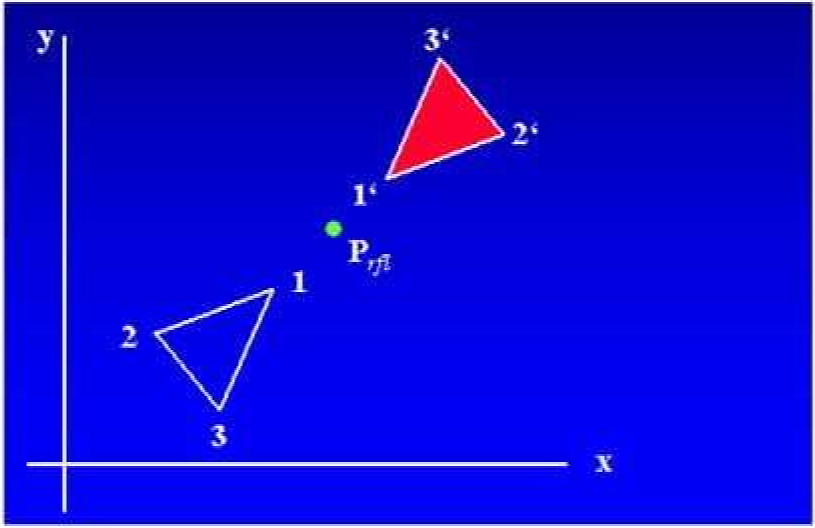
- Reflection relative to the coordinate origin



$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

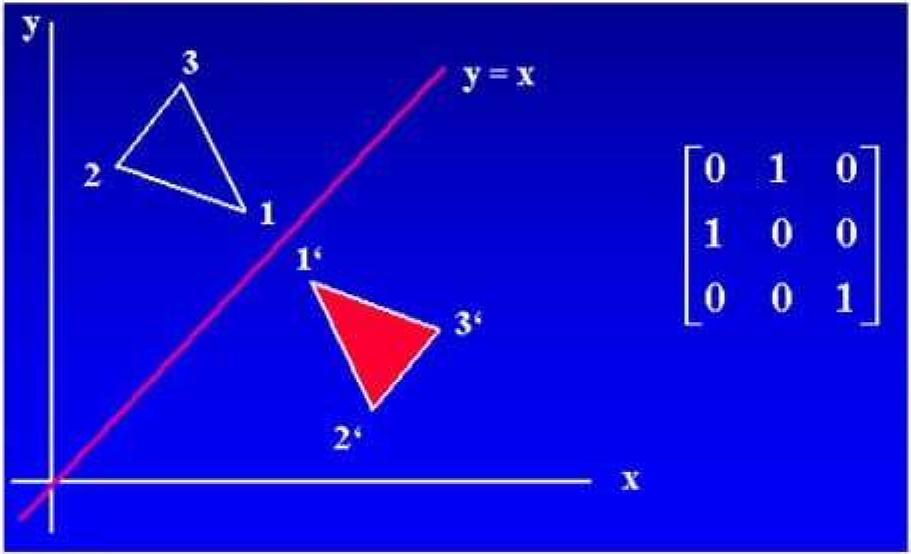
Reflection

- Reflection of an object relative to an axis perpendicular to the xy plane through P_{rfl}



Reflection

- Reflection about the line y = x



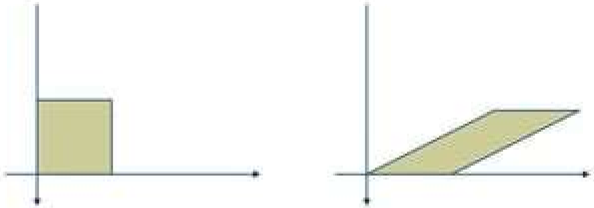
Shear

- The x-direction shear relative to x axis

$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x' = x + sh_x \cdot y$$
$$y' = y$$

If $sh_x = 2$:



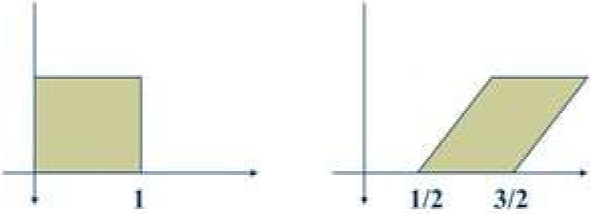
Shear

- **The x-direction shear relative to y = yref**

$$\begin{bmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x' = x + sh_x \cdot (y - y_{ref})$$
$$y' = y$$

If $sh_x = ½$ $y_{ref} = -1$:



Shear

- The y-direction shear relative to $x = x_{ref}$

$$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix}$$

$$x' = x$$
$$y' = sh_y(x - x_{ref}) + y$$

If $sh_y = ½$ $x_{ref} = -1$: