



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore-641035.



Green's Theorem:
Statement:

If R is the closed region of (x, y) bounded by a simple closed curve C , If (M, N) are continuous function of (x, y) having continuous derivatives w.r.t. x & y , i.e., single integral is

$$\int \int_R (M dx + N dy) = \int \int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Example: $\int \int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

Verify green theorem for $\int (xy + y^2) dx + x^2 dy$ where x & y is closed region bounded by $y = x^2$, $y = x$.

Soln:

$$= \int \int_R (x^2 - x^4) dx dy$$



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UNIT-I Vector calculus

Green's theorem

Given, $\int_C (xy + y^2) dx + x dy$

Let $R = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x^2\}$

$y = x^2 \rightarrow \text{Eqn } ①$
 $y = x \rightarrow \text{Eqn } ②$

Sub $y = x$ in $①$
 $x = x^2$
 $x^2 - x = 0$
 $x(x-1) = 0$
 $x=0 \cup x=1$

If $x=0$ then $y=0$
If $x=1$ then $y=1$
 $x=0 \quad x=1$
 $\frac{\partial N}{\partial x} = 2y \quad \frac{\partial M}{\partial y} = x$

RHS : $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$= \int_0^1 \int_{x^2}^x (2x - x - 2y) dx dy$

$= \int_0^1 \int_{x^2}^x (x - 2y) dy dx$

$= \int_0^1 \left[\int_{x^2}^x x dy - \int_{x^2}^x 2y dy \right] dx$

$= \int_0^1 \left[(xy) \Big|_{x^2}^x - 2 \left[\frac{y^2}{2} \right] \Big|_{x^2}^x \right] dx$

$= \int_0^1 \left[(x^2 - x^3) - 2 \left[\frac{x^2 - x^4}{2} \right] \right] dx$

$= \int_0^1 \left[(x^2 - x^3) - \left[\frac{x^2 - x^4}{2} \right] \right] dx$

$= \int_0^1 (x^2 - x^3 - x^2 + x^4) dx$

$= \int_0^1 (x^4 - x^3) dx$



$$\begin{aligned}
 &= \int_0^1 x^4 dx - \int_0^1 x^3 dx \\
 &= \left[\frac{x^5}{5} \right]_0^1 - \left[\frac{x^4}{4} \right]_0^1 \\
 &= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} \\
 &= -\frac{1}{20}.
 \end{aligned}$$

LHS :

$$\begin{aligned}
 \int_C (xy + y^2) dx + x^2 dy &= \int_{OA} (xy + y^2) dx + x^2 dy + \\
 &\quad \int_{AO} (xy + y^2) dx + x^2 dy
 \end{aligned}$$

Along OA, $y = 0$, $x \in [0, 1]$

$$\begin{aligned}
 \frac{dy}{dx} &= 0 \\
 dy &= 0 dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 (x \cdot 0 + 0^2) dx + x^2 (2x) dx \\
 &= \int_0^1 (x^3 + x^4) dx + 2x^3 dx
 \end{aligned}$$

Integrating both sides:

$$\begin{aligned}
 &= \int_0^1 (x^3 + x^4) dx + \int_0^1 2x^3 dx \\
 &= \left[\frac{x^4}{4} + \frac{x^5}{5} \right]_0^1 + \left[\frac{2x^4}{4} \right]_0^1 \\
 &= \left[\frac{1}{4} + \frac{1}{5} \right] + \frac{1}{2} \\
 &= \frac{9}{20} + \frac{1}{2} = \frac{19}{20}
 \end{aligned}$$

Along AO, $x = 0$, $y \in [0, 1]$

$$\begin{aligned}
 \frac{dx}{dy} &= 0 \\
 dx &= 0 dy
 \end{aligned}$$



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$$\begin{aligned}
 & \int_{AO} (xy + y^2) dx + \int_{OB} x^2 dy = \int_0^b \left[xy + \frac{y^3}{3} \right]_0^b dx = \int_0^b \left(b^2 x + \frac{b^3}{3} \right) dx \\
 & = \int_0^b (x^2 + x^2 + x^2) dx = \int_0^b (3x^2) dx = \left[\frac{x^3}{3} \right]_0^b = \frac{b^3}{3} \\
 & = \int_{OA} 3x^2 dx = \left[\frac{3x^3}{3} \right]_0^b = b^3 \\
 & \therefore \int_{AO} (xy + y^2) dx + \int_{OB} x^2 dy = pb^3 + xb^3(p+pb) \quad (LHS) \\
 & \int_{OA} (xy + y^2) dx = -1 \int_{OA} (p+pb) dy \\
 & \text{Now, } LHS = \int_{AO} (xy + y^2) dx + \int_{OB} x^2 dy + \int_{OA} (xy + y^2) dx + \int_{OB} x^2 dy \\
 & = \frac{19}{20} + (-1) \int_{AO} (p+pb) dy = pb^3 \\
 & = \frac{19}{20} - pb^3 = -\frac{1}{20}(p+pb) \\
 & \therefore LHS = RHS \quad \therefore \text{Hence proved.}
 \end{aligned}$$