Method of variation of Parameter:

An equ which is in the form of $\frac{d^2y}{dx^2} + P\cdot\frac{dy}{dx} + Qy = R(x)$, then we can solve by using method of variation of Parameter, where P, Q, R are fn.

Methods to find solutions:

Step 1: Find the C.F of the given differential equation from this find y_1 and y_2

Step 2: Find $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$ where w is wronskian.

Step 3: To find particular integral.

$P.I = Ay_1 + By_2$

where $A = -\int \frac{R(x)}{w} y_2\, dx$

$B = \int \frac{R(x)}{w} \cdot y_1 \cdot dx$

Step 4: The General solution is $y = C.F + P.I$

Notes:

1) $\int \cot x \cdot dx = \log(\sin x)$

2) $\int \tan x \cdot dx = \log(\sec x)$

3) $\int \operatorname{cosec} x \cdot dx = -\log(\operatorname{cosec} x + \cot x)$

4) $\int \sec x \cdot dx = \log(\sec x + \tan x)$

Example: 1

Solve $\frac{d^2y}{dx^2} + 4y = 4(\tan 2x)$

Soln: $(D^2+4)y = 4(\tan 2x)$

The A.E is $m^2 + 4 = 0$

$m^2 = -4$

$m^2 = i^2 2^2$

$m = \pm 2i$

$\alpha = 0,\ \beta = 2$

$C.F = e^{0x}[A\cos 2x + B\sin 2x] = A\cos 2x + B\sin 2x$

$y_1 = \cos 2x$ | $y_2 = \sin 2x$

$y_1' = -2\sin 2x$ | $y_2' = 2\cos 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= | 2\cos^2 2x + 2\sin^2 2x |$$

$$2(\cos^2 2x + \sin^2 2x) \qquad \therefore \boxed{W = 2}$$

$$P.I = Ay_1 + By_2$$

$$A = -\int \frac{R(x)}{W} y_2 \, dx$$

$$= -\int \frac{4\tan 2x}{2} \sin 2x \, dx = -2\int \tan 2x \cdot \sin 2x \, dx$$

$$= -2 \int \frac{\sin^2 2x}{\cos 2x} \cdot dx = -2 \int \frac{1-\cos^2 2x}{\cos 2x} \cdot dx$$

$$= -2\left[\int \frac{1}{\cos 2x} \cdot dx - \int \frac{\cos^2 2x}{\cos 2x} \cdot dx\right]$$

$$= -2\left[\int \sec 2x \, dx - \int \cos 2x \cdot dx\right]$$

$$= -2\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$

$$A = \sin 2x - \log(\sec 2x + \tan 2x)$$

$$B = \int \frac{R(x)}{W} y_1 \, dx$$

$$= \int \frac{4\tan 2x}{2} \cos 2x \cdot dx$$

$$= \int 2 \frac{\sin 2x}{\cos 2x} \cdot \cos 2x \cdot dx$$

$$= 2\int \sin 2x \cdot dx$$

$$B = -\frac{\cos 2x}{2} = -\cos 2x$$

$$P.I = Ay_1 + By_2$$

$$= (\sin 2x - \log(\sec 2x + \tan 2x))\cos 2x - (\cos 2x)(\sin 2x)$$

$$y = C.F + P.I$$

$$y = (\cos 2x + \sin 2x + \sin 2x - \log(\sec 2x + \tan 2x)\cos 2x - (\cos 2x)(\sin 2x).$$

Example : 2

$(D^2+1)y = \sec x$

Soln: Given

$$(D^2+1)y = \sec x$$

The A.E is $m^2+1=0$

$$m^2 = -1$$

$$m = \pm i$$

$C.F = e^{0x}[A\cos x + B\sin x]$

$P.I = Pf_1 + Qf_2 \quad Ay_1 + By_2$

$P = -\int \frac{R(x)}{W} f_2\, dx$

$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} \quad f_1 = \cos x \quad f_2 = \sin x$

$f_1' = -\sin x \quad f_2' = \cos x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$P = -\int \frac{\sec x}{1} \sin x\, dx$

$= -\int \frac{\sin x}{\cos x} dx$ $\quad \frac{R(x)}{W} y_1$

$= -\int \tan x$

$P = -\log(\sec x)$

$Q(x) = \int x^n dx = \frac{x^{n+1}}{n+1} = \frac{x^1}{1}$

$Q = \int \frac{\sec x}{1} \cos x\, dx$

$= \int \frac{1}{\cos x} \cdot \cos x\, dx$

$Q = \int dx = x$

$P.I = Pf_1 + Qf_2$

$= -\log(\sec x)\cos x + x\sin x$

$y = C.F + P.I$

$= A\cos x + B\sin x - \log(\sec x)\cos x + x\sin x.$

Example : 3

$(D^2+1)y = \operatorname{cosec} x.$

Soln: Given, $(D^2+1)y = \operatorname{cosec} x.$

The A.E is $m^2+1 = 0$

$m = \pm i$

$\alpha = 0,\ \beta = 1$

$\therefore C.F = e^{0x}[A\cos x + B\sin x]$

$P.I = Ay_1 + By_2$

$A = -\int \frac{R(x)}{W} y_2\, dx$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x = 1$

$$A = -\int \frac{\operatorname{cosec} x}{1} \cos x \, dx.$$

$$= -\int \frac{1}{\sin x} \cos x \, dx$$

$$= -\int \cot x \, dx$$

$$= -\log(\sin x)$$

$$B = \int \frac{R(x)}{W} y_1 \, dx$$

$$= \int \frac{\operatorname{cosec} x}{1} \cdot \sin x \, dx$$

$$= \int \frac{1}{\sin x} \sin x \, dx$$

$$= \int dx$$

$$B = x$$

$$P.I = A y_1 + B y_2$$

$$= -\log(\sin x)\cos x + x \sin x$$

$$y = C.F + P.I$$

$$= e^{0x}[A\cos x + B\sin x] - \log(\sin x)$$