



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore-641035.



UNIT-III COMPLEX DIFFERENTIATION

HARMONIC FUNCTION

Laplace equation :

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called Laplace equation.

Harmonic equation :

Any function with 2 variables having its 2nd order partial derivatives which satisfies Laplace eqn. is called a harmonic eqn.

Conjugate Harmonic function :

If u and v are harmonic functions such that $u+iv$ is analytic, then each is called the conjugate harmonic function of the other.

Here u is conjugate harmonic of v and v is conjugate harmonic of u .

Q1. Prove that $u = e^x \cos y$ is harmonic.

Soln.

Given $u = e^x \cos y$

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y & \frac{\partial v}{\partial y} &= -e^x \sin y \\ \frac{\partial^2 u}{\partial x^2} &= e^x \cos y & \frac{\partial^2 v}{\partial y^2} &= -e^x \cos y \end{aligned}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$$

Hence u satisfies Laplace equation.

∴ The function u is harmonic.

Q2. Prove that $u = \frac{1}{2} \log(x^2 + y^2)$ is harmonic.



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Soln.

$$\text{Given } u = \frac{1}{2} \log(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} (2x)$$

$$= \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)^2}$$

$$= \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$= \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} (2y)$$

$$= \frac{y}{y^2 + x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$= \frac{-x^2 + y^2 + x^2 - y^2}{(x^2 + y^2)^2}$$

Hence u satisfies Laplace eqn.

$\therefore u$ is harmonic