(14) Find the values of a and b so that the surface $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at $(2,-1,-3)$.

Soln:

$$a = -7/3 \ \& \ b = 64/9$$

DIVERGENCE OF A VECTOR POINT FUNCTION:

Let $\vec{F}$ be any given continuously differentiable vector point function then the divergence of $\vec{F}$ is defined as,

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{F}$$

$$= \vec{i}\cdot\frac{\partial \vec{F}}{\partial x} + \vec{j}\cdot\frac{\partial \vec{F}}{\partial y} + \vec{k}\cdot\frac{\partial \vec{F}}{\partial z}$$

Note :

1. $\nabla\cdot\vec{F}$ is a scalar point function.
2. If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a continuously differentiable vector point function then,

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Solenoidal vector :

A vector $\vec{F}$ is said to be solenoidal vector if $\text{div}\,\vec{F} = 0$.

CURL OF A VECTOR POINT FUNCTION:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be any given continuously differentiable vector point function, the curl or rotation of $\vec{F}$ is defined as,

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{F}$$

$$= \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note: $\nabla \times \vec{F}$ is a vector point function.

IRROTATIONAL VECTOR:

A vector $\vec{F}$ is said to be irrotational if $\nabla \times \vec{F} = 0$

i.e., $\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$

CONSERVATIVE VECTOR FIELD:

If a vector point function $\vec{F}$ is expressible as the gradient of a scalar point function φ, then $\vec{F}$ is conservative i.e., $\vec{F}$ is conservative if $\vec{F} = \nabla\varphi$. Here φ is called scalar potential.

$\vec{F}$ is conservative force if $\text{curl } \vec{F} = 0$.

PROBLEMS :

(1) Prove that $\operatorname{curl}(\nabla\varphi) = 0$ (or) $\nabla \times \nabla\varphi = 0$.

Soln:

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\operatorname{Curl}(\nabla\varphi) = \nabla\times\nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial y\,\partial z}\right) - \vec{j}\left(\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial^2\varphi}{\partial x\,\partial z}\right) + \vec{k}\left(\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial x\,\partial y}\right)$$

$$= 0$$

(2) Prove that $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ (or) $\nabla\cdot(\nabla\times\vec{F}) = 0$

Soln:

$$\operatorname{Curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \quad \text{if } \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\nabla\times\vec{F} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla\cdot\nabla\times\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

(7) Find the constants a, b, c so that the vector

$\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$

is irrotational.

Soln:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2).$$

Given : $\vec{F}$ is irrotational.

i.e., $\nabla \times \vec{F} = 0$.

$\vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0.$

$c+1 = 0 \Rightarrow c = -1$

$4 - a = 0 \Rightarrow a = 4$

$b - 2 = 0 \Rightarrow b = 2$

(8) Find 'a' so that the vector

$\vec{A} = (ax^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ is irrotational.

Soln:

Given : $\vec{A}$ is irrotational.

$$\nabla \times \vec{A} = 0.$$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ax^2 - y^2 + x & -(2xy+y) & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(-2y + 2y) = 0.$$

$\therefore$ 'a' is arbitrary.

(9) Prove $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential φ such that $\vec{F} = \nabla\varphi$.

Soln:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= 0$$

$\therefore \vec{F}$ is irrotational.

To find φ:

$$\nabla\varphi = (y^2\cos x + z^2)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$$

We know that $\nabla\varphi = \vec{i}\dfrac{\partial\varphi}{\partial x} + \vec{j}\dfrac{\partial\varphi}{\partial y} + \vec{k}\dfrac{\partial\varphi}{\partial z}$

$$\frac{\partial \varphi}{\partial x} = y^2 \cos x + z^3 \Rightarrow \varphi = y^2 \sin x + z^3 x + f(y,z)$$

$$\frac{\partial \varphi}{\partial y} = 2y \sin x - 4 \Rightarrow \varphi = y^2 \sin x - 4y + f(x,z)$$

$$\frac{\partial \varphi}{\partial z} = 3xz^2 \Rightarrow \varphi = \cancel{y^2}\, x z^3 + f(x,y)$$

(10) Show that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational. Find φ such that $\vec{F} = \nabla\varphi$.

Soln: $\varphi = 3x^2y + xz^3 - yz + c$

(11) If $\nabla\varphi = yz\vec{i} + xz\vec{j} + xy\vec{k}$ then find φ.

Soln: $\varphi = xyz + c$

(12) Prove that $\operatorname{div} \hat{r} = 2/r$.

Soln:

$$\operatorname{div} \hat{r} = \nabla \cdot \left(\frac{\vec{r}}{r}\right)$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r} - \frac{1}{r^2} \cdot x\frac{\partial r}{\partial x} + \frac{1}{r} - \frac{1}{r^2} y \frac{\partial r}{\partial y} + \frac{1}{r} - \frac{1}{r^2} z \frac{\partial r}{\partial z}$$

$$= \frac{3}{r} - \frac{1}{r^2}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

Now $r^2 = x^2 + y^2 + z^2$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$