



## DEPARTMENT OF MATHEMATICS

### GREEN'S THEOREM IN A PLANE:

If  $R$  is a closed region of the  $xy$ -plane bounded by a simple closed curve  $C$  and if  $M$  and  $N$  are continuous functions of  $x$  and  $y$  having continuous derivatives in  $R$  then

$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Where  $C$  is a curve traversed in the anticlockwise direction.

### PROBLEMS:

- ① Evaluate by Green's theorem  $\int_C (xy + x^2) dx + (x^2 + y^3) dy$  where  $C$  is the square formed by  $x = -1$ ,  $x = 1$ ,  $y = -1$ ,  $y = 1$ .

Solution:



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Let  $R$  be the region enclosed by  $C$ .

By Green's theorem,

$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Here  $M = xy + x^2 \Rightarrow \frac{\partial M}{\partial y} = x$

$N = x^2 + y^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$

$$\int_C (xy + x^2) dx + (x^2 + y^2) dy = \iint_R (2x - x) dx dy$$

$$= \int_{-1}^1 \int_{-1}^1 x dx dy$$

$$= \int_{-1}^1 \left[ \frac{x^2}{2} \right]_{-1}^1 dy = 0$$

② Evaluate by Green's theorem  $\int_C e^{-x} (\sin y dx + \cos y dy)$

where  $C$  is the rectangle with vertices  $(0,0)$ ,  $(\pi,0)$ ,  $(\pi, \pi/2)$ ,  $(0, \pi/2)$ .

Soln: Let  $R$  be the region enclosed by  $C$ .

By Green's theorem,

$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Here  $M = e^{-x} \sin y \Rightarrow \frac{\partial M}{\partial y} = e^{-x} \cos y$

$N = e^{-x} \cos y \Rightarrow \frac{\partial N}{\partial x} = -e^{-x} \cos y$

$$\therefore \int_C e^{-x} (\sin y dx + \cos y dy) = \iint_R (-e^{-x} \cos y - e^{-x} \cos y) dx dy$$



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$$\begin{aligned}
 &= \int_0^{\pi/2} \int_0^{\pi} (-2e^{-x} \cos y) dx dy \\
 &= -2 \int_0^{\pi/2} \int_0^{\pi} e^{-x} \cos y dx dy \\
 &= 2(e^{-\pi} - 1).
 \end{aligned}$$

③ Evaluate by Green's theorem

$\int_C (x^2 - \cosh y) dx + (y + \sin x) dy$  where  $C$  is the rectangle with vertices  $(0, 0)$ ,  $(\pi, 0)$ ,  $(\pi, 1)$ ,  $(0, 1)$ .

Soln:  $\pi(\cosh 1 - 1)$ .

④ Verify Green's theorem in the plane for

$\int_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$  where  $C$  is the boundary of the region defined by  $x = y^2$ ,  $y = x^2$ .

Soln:

By Green's theorem,

$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\text{Given: } \int_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$$

$$\text{Here } M = 3x^2 - 8y^2, \quad N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y$$

Step 1:

$$\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R (-6y + 16y) dx dy$$





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$$\begin{aligned}
 &= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y \, dx \, dy \\
 &= \int_0^1 10y \left[ x \right]_{y^2}^{\sqrt{y}} dy = \int_0^1 10y (\sqrt{y} - y^2) dy \\
 &= 10 \int_0^1 (y^{3/2} - y^3) dy = 10 \left[ \frac{y^{5/2}}{5/2} - \frac{y^4}{4} \right]_0^1 \\
 &= 10 \left( \frac{2}{5} - \frac{1}{4} \right) = 10 \left( \frac{8-5}{20} \right) = \frac{3}{2} \rightarrow \textcircled{1}
 \end{aligned}$$

Step 2 :

To evaluate  $\int_C Mdx + Ndy$  we take  $C$  in different paths

(i) along  $OA$  ( $y=x^2$ )

(ii) along  $AD$  ( $x=y^2$ )

(i) Along  $OA$  :

$$\begin{aligned}
 \int_{OA} Mdx + Ndy &= \int_{OA} [3x^2 - 8x^4] dx + [4x^2 - 6x \cdot x^2] 2x dx \\
 &= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4) dx \quad (\because x^2 = y, 2x dx = dy) \\
 &= \int_0^1 (-20x^4 + 8x^3 + 3x^2) dx \quad (\because \text{Along } OA, x \text{ varies from } 0 \text{ to } 1) \\
 &= \left[ -20 \frac{x^5}{5} + \frac{8x^4}{4} + \frac{3x^3}{3} \right]_0^1 = -1
 \end{aligned}$$

**DEPARTMENT OF MATHEMATICS**(ii) Along  $A_0$  :

$$\int_{A_0} M dx + N dy = \int_{A_0} (3y^4 - 8y^3) 2y dy + (4y - 6y^2) dy$$

$$(\because y^2 = x, 2y dy = dx)$$

$$= \int_{A_0} (6y^5 - 16y^3 + 4y - 6y^3) dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y) dy \quad (\because \text{Along } A_0 \text{ } y \text{ varies from 1 to 0}).$$

$$= \left[ \frac{6y^6}{6} - \frac{22y^4}{4} + \frac{4y^2}{2} \right]_1^0$$

$$= 5/2.$$

$$\therefore \int_C M dx + N dy = \int_{OA} M dx + N dy + \int_{A_0} M dx + N dy$$

$$= -1 + \frac{5}{2}$$

$$= \frac{3}{2} \quad \rightarrow \textcircled{2}$$

$$\textcircled{1} = \textcircled{2}$$

Hence Green's theorem is verified.