STOKE'S THEOREM:

If $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve C then,

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds}$$

Where $\hat{n}$ is the unit normal vector at any point of S.

Note:

1. If $\vec{F}$ is irrotational, $\nabla\times\vec{F} = 0$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = 0$$ & hence $\vec{F}$ is Conservative.

2. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

PROBLEMS:

(1) Verify stoke's theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\,\vec{j}$ in the rectangular region in the xoy plane bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$.

Soln:

By Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$

RHS :

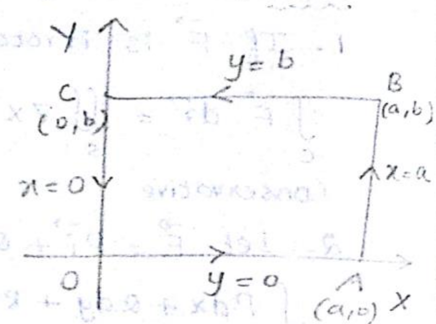
Given : $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y+2y) = 4y\vec{k}$$

Here the surface S denotes the rectangle OABC and the unit outward normal vector is $\vec{k}$.

i.e., $\hat{n} = \vec{k}$



$\therefore$ curl $\vec{F} \cdot \hat{n}\, ds$

$= 4y\vec{k} \cdot \vec{k}\, dx\,dy$

$= 4y\, dx\, dy$

$$\therefore \iint \nabla \times \vec{F} \cdot \hat{n}\, ds = \iint_S 4y\, dx\, dy$$

$$= \int_0^b \int_0^a 4y\, dx\, dy$$

$$= 4\int_0^b [x]_0^a\, y\, dy$$

$$= 4a\left[\frac{y^2}{2}\right]_0^b$$

$$= \frac{4ab^2}{2}$$

$$= 2ab^2 \longrightarrow (1)$$

LHS :

$$\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F}\cdot d\vec{r} = (x^2-y^2)\,dx + 2xy\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(x^2-y^2)\,dx + 2xy\,dy]$$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA $(y=0)$:

$$\int_{OA} (x^2-y^2)\,dx + 2xy\,dy = \int_0^a x^2\,dx \quad \left[\begin{array}{l} \because y=0 \\ \Rightarrow dy=0 \\ \text{Along } OA, \\ x \to 0 \text{ to } a \end{array}\right]$$

$$= \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3}$$

Along AB $(x=a)$:

$$\int_{AB} (x^2-y^2)\,dx + 2xy\,dy = \int_0^b 2ay\,dy \quad \left[\begin{array}{l} \because x=0 \\ \Rightarrow dx=0 \\ \text{Along } AB \\ y \to 0 \text{ to } b \end{array}\right]$$

$$= 2a\left[\frac{y^2}{2}\right]_0^b$$

$$= ab^2$$

Along BC $(y=b)$:

$$\int_{BC} (x^2-y^2)\,dx + 2xy\,dy = \int_a^0 (x^2-b^2)\,dx \quad \left[\begin{array}{l} \because y=b \\ \Rightarrow dy=0 \\ \text{Along } BC \\ x \to a \text{ to } 0 \end{array}\right]$$

$$= \left[\frac{x^3}{3} - b^2x\right]_a^0$$

$$= -\frac{a^3}{3} + ab^2$$

Along $CO\ (x=0)$:

$$\int_{CO} (x^2-y^2)\,dx + 2xy\,dy = \int_{CO} 0 + 0 \qquad (\because x=0,\ dx=0)$$

Hence $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

$$= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$= 2ab^2 \rightarrow (2)$$

From (1) & (2),

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$

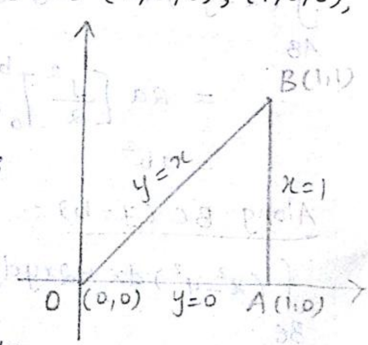
Hence Stoke's theorem is verified.

(2) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and C is the boundary of the triangle with vertices at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$.

Soln:

By Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$



Since z coordinate is zero in all the three vertices of the given triangle, the triangle lies on the xy plane.

$$\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$$