

**DEPARTMENT OF MATHEMATICS**STOKE'S THEOREM:

If \vec{F} is any continuous differentiable vector function and S is a surface enclosed by a curve C then,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$

Where \hat{n} is the unit normal vector at any point of S .

Note:

1. If \vec{F} is irrotational, $\nabla \times \vec{F} = 0$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = 0 \quad \& \quad \text{hence } \vec{F} \text{ is}$$

Conservative.

2. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\begin{aligned} \int_C P dx + Q dy + R dz &= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \, dz + \\ &\quad \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \, dx \\ &\quad + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy \end{aligned}$$

PROBLEMS:

- ① Verify Stoke's theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$.

Soln:

By Stoke's theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$



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RHS :

Given: $\vec{F} = (x^2 - y^2) \vec{i} + 2xy \vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y + 2y) = 4y \vec{k}$$

Here the surface S denotes the rectangle $OACB$ and the unit outward normal vector is \vec{k} .

i.e., $\hat{n} = \vec{k}$

$$\therefore \text{curl } \vec{F} \cdot \hat{n} \, ds = 4y \vec{k} \cdot \vec{k} \, dx \, dy$$

$$= 4y \, dx \, dy$$

$$\therefore \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = \iint_S 4y \, dx \, dy$$

$$= \int_0^b \int_0^a 4y \, dx \, dy$$

$$= 4 \int_0^b [x]_0^a y \, dy$$

$$= 4a \left[\frac{y^2}{2} \right]_0^b$$

$$= \frac{4ab^2}{2}$$

$$= 2ab^2 \longrightarrow \textcircled{1}$$



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LHS :

$$\vec{F} = (x^2 - y^2) \vec{i} + 2xy \vec{j}$$

$$d\vec{r} = dx \vec{i} + dy \vec{j}$$

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2) dx + 2xy dy$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C [(x^2 - y^2) dx + 2xy dy]$$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA (y = 0) :

$$\begin{aligned} \int_{OA} (x^2 - y^2) dx + 2xy dy &= \int_0^a x^2 dx \quad \left[\begin{array}{l} \because y=0 \\ \Rightarrow dy=0 \\ \text{Along OA,} \\ x \rightarrow 0 \text{ to } a \end{array} \right] \\ &= \left[\frac{x^3}{3} \right]_0^a \\ &= \frac{a^3}{3} \end{aligned}$$

Along AB (x = a) :

$$\begin{aligned} \int_{AB} (x^2 - y^2) dx + 2xy dy &= \int_0^b 2ay dy \quad \left[\begin{array}{l} \because x=a \\ \Rightarrow dx=0 \\ \text{Along AB} \\ y \rightarrow 0 \text{ to } b \end{array} \right] \\ &= 2a \left[\frac{y^2}{2} \right]_0^b \\ &= ab^2 \end{aligned}$$

Along BC (y = b) :

$$\begin{aligned} \int_{BC} (x^2 - y^2) dx + 2xy dy &= \int_a^0 (x^2 - b^2) dx \quad \left[\begin{array}{l} \because y=b \\ \Rightarrow dy=0 \\ \text{Along BC} \\ x \rightarrow a \text{ to } 0 \end{array} \right] \\ &= \left[\frac{x^3}{3} - b^2 x \right]_a^0 \\ &= -\frac{a^3}{3} + ab^2 \end{aligned}$$



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Along $CO (x=0)$:

$$\int_{CO} (x^2 - y^2) dx + 2xy dy = \int_{CO} 0 + 0 \quad (\because x=0, dx=0)$$

$$\begin{aligned} \text{Hence } \int_C \vec{F} \cdot d\vec{r} &= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} \\ &= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 \\ &= 2ab^2 \rightarrow (2) \end{aligned}$$

From (1) & (2),

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} ds$$

Hence Stoke's theorem is verified.

(2) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by Stoke's theorem where

$\vec{F} = y^2 \vec{i} + x^2 \vec{j} - (x+z) \vec{k}$ and C is the boundary of the triangle with vertices at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$.

Soln:

By Stoke's theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} ds$$

Since z coordinate is zero in all the three

vertices of the given triangle, the triangle lies on the xy plane.

$$\vec{F} = y^2 \vec{i} + x^2 \vec{j} - (x+z) \vec{k}$$

