GAUSS DIVERGENCE THEOREM:

If $\vec{F}$ is a vector point function, finite and differentiable in a region R bounded by a closed surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv}$$

where $\hat{n}$ is the unit vector in the positive (outward drawn) normal to S.

Problems:

(1) Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube $x=0, x=1, y=0, y=1, z=0, z=1$.

Soln: By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

RHS:

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot(4xz\vec{i} - y^2\vec{j} + yz\vec{k})$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

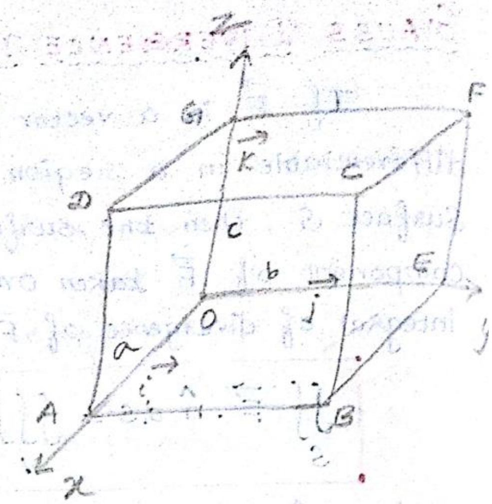
$$= 4z - y$$

$$\iiint_V \nabla\cdot\vec{F}\, dv = \int_0^1\int_0^1\int_0^1 (4z-y)\, dx\, dy\, dz = \frac{3}{2} \longrightarrow (1)$$

LHS:

$$\iint_S \vec{F}\cdot\hat{n}\,ds$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$



Surface	$\hat{n}$	ds	Face equation
S_1 - ABCD	$\vec{i}$	$dy\,dz$	$x=1$
S_2 - OEFG	$-\vec{i}$	$dy\,dz$	$x=0$
S_3 - BCEF	$\vec{j}$	$dx\,dz$	$y=1$
S_4 - OADG	$-\vec{j}$	$dx\,dz$	$y=0$
S_5 - DCGF	$\vec{K}$	$dx\,dy$	$z=1$
S_6 - OABE	$-\vec{K}$	$dx\,dy$	$z=0$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{ABCD} (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{K})\cdot\vec{i}\,dy\,dz$$

$$= \int_0^1\int_0^1 4xz\,dy\,dz$$

$$= \int_0^1\int_0^1 4z\,dy\,dz \quad (\because x=1)$$

$$= 2$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{OEFG} (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{K})\cdot(-\vec{i})\,dy\,dz$$

$$= \int_0^1\int_0^1 (-4xz)\, dy\, dz = 0 \quad (\because x = 0)$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = \iint_{BCEF} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{j}\, dx\, dz$$

$$= \int_0^1\int_0^1 (-y^2)\, dx\, dz \quad (\text{Here } y = 1)$$

$$= \int_0^1\int_0^1 -dx\, dz = -1$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \iint_{OADG} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{j})\, dx\, dz$$

$$= \int_0^1\int_0^1 y^2\, dx\, dz = 0 \quad (\because y = 0)$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds = \iint_{DCGF} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{k}\, dx\, dy$$

$$= \int_0^1\int_0^1 yz\, dx\, dy = \int_0^1\int_0^1 y\, dx\, dy \quad (\because z = 1)$$

$$= \int_0^1 y\left[x\right]_0^1 dy = \int_0^1 y\, dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\, ds = \iint_{OABE} (-yz)\, dx\, dy = 0 \quad (\because z = 0)$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$= 2 + 0 + (-1) + 0 + 1/2 + 0$$

$$= 3/2 \longrightarrow (2)$$

From (1) & (2),

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

(2) Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

(or)

Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ and S is the surface of the rectangular parallelopiped bounded by $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, $z=c$.

Soln :

By Gauss - divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv.$$

RHS :

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left((x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= (2x + 2y + 2z)$$

$$= 2(x + y + z).$$

$$\iiint_V \nabla \cdot \vec{F}\, dv = \int_0^a \int_0^b \int_0^c 2(x + y + z)\, dx\, dy\, dz$$

$$= 2\int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2}\right]_0^c dx\, dy$$

$$= 2\int_0^a \int_0^b \left(xc + yc + \frac{c^2}{2}\right) dx\, dy$$