3. Solution of Equations

Newton Method (or) Newton Raphson Method

Netwon Raphson method is extensively used for analysis of flow in water distribution networks. It is used to find the roots of non linear equations.

Formula:

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}, \text{ Provided } F'(x_n) \neq 0.$$

order = 2.

Newton Raphson condition:

$$|F(x)F''(x)| < |F'(x)|^2.$$

Problems:

1. Find the smallest positive root of the eqn $x^3 - 2x + 0.5 = 0$.

Let $F(x) = x^3 - 2x + 0.5$

Now $F'(x) = 3x^2 - 2$

$F(0) = 0.5$

$F(1) = -0.5$

$\therefore$ The ~~p~~ root lies b/w 0 & 1

Since $|F(0)| = |F(1)|$

Let us assume $x_0 = 0$..

Newton Raphson Formula :-

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

Put $n=0$, $x_1 = x_0 - \frac{F(x_0)}{F'(x_0)} = 0 - \frac{0.5}{-2} = 0.25$

Put $n=1$, $x_2 = x_1 - \frac{F(x_1)}{F'(x_1)} = 0.25 - \frac{F(0.25)}{F'(0.25)}$

$$x_2 = 0.2586$$

Put $n=2$, $x_3 = x_2 - \frac{F(x_2)}{F'(x_2)}$

$$= 0.2586.$$

Since x_2 & x_3 are equal roots, the smallest positive root is 0.2586.

d) Compute the real root of $x \log x = 1.2$ correct to 3 decimal places using Newton Raphson method.

Let $f(x) = x \log x - 1.2$

$$F'(x) = x\left(\frac{1}{x}\right) + \log x(1) - 0 = 1 + \log x$$

Now $F(0) = -ve$

$F(1) = -1.2$

$F(2) = -0.5979$ $\therefore$ The root lies between

$F(3) = 0.2314$ 2 & 3

Since $|F(2)| > |F(3)|$. Let us assume $x_0 = 3$

Newton - Raphson formula,

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)} = 2.8434$$

$$x_2 = 2.8434 - \frac{F(2.8434)}{F'(2.8434)}$$

$$= 2.7822$$

$$x_3 = 2.7822 - \frac{F(2.7822)}{F'(2.7822)}$$

$$= 2.7576$$

llly,
$x_4 = 2.7476$

$x_5 = 2.7435$

$x_6 = 2.7418$

$x_7 = 2.7411$

$x_8 = 2.7408$

$x_9 = 2.7407$

$x_{10} = 2.7407$

$\therefore$ The required root is 2.7407

HW find the +ve root of $2x^3 - 3x - 6 = 0$

Ans: 1.7838

3. Find the -ve root of $x^3 - \sin x + 1 = 0$.

[radian mode]

Let $F(x) = x^3 - \sin x + 1$

$$F'(x) = 3x^2 - \cos x$$

$$F(0) = 1$$

$$F(-1) = 0.8415 \ (+ve)$$

$$F(-2) = -6.0907 \ (-ve)$$

$\therefore$ The root lies between -1 & -2

Since $|F(-1)| < |F(-2)|$. Let us assume $x_0 = -1$

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

$$x_1 = -1.3421$$

$$x_2 = -1.3421 - \frac{F(-1.3421)}{F'(-1.3421)}$$

$$= -1.2564$$

$$x_3 = -1.2491$$

$$x_4 = -1.2491$$

Since x_3 & x_4 are equal

$\therefore$ The required -ve root is -1.2491

(4) obtain Newton's Iterative formula for finding $\sqrt{N}$ where N is a +ve real no. Hence evaluate $\sqrt{5}$

Let $x = \sqrt{N} \Rightarrow x^2 = N \Rightarrow x^2 - N = 0$.

$F(x) = x^2 - N$

$F'(x) = 2x$

Now $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$

$= x_n - \frac{x_n^2 - N}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$

$= \frac{x_n^2 + N}{2x_n}$, which is an iterative formula for $\sqrt{N}$

To find $\sqrt{5}$

$x = \sqrt{5}$

$x^2 - 5 = 0$

$F(x) = x^2 - 5$; $F'(x) = 2x$.

$F(0) = -5$ (-ve)

$F(1) = 1 - 5 = -4$

$F(2) = 4 - 5 = -1$ (-ve)

$F(3) = 9 - 5 = 4$ (+ve)

$\therefore$ The root lies between 2 & 3

$\because |F(2)| < |F(3)|$, let us assume that $x_0 = 2$

Now, $x_{n+1} = \frac{x_n^2 + N}{2x_n} \Rightarrow n=0 \quad x_1 = \frac{x_0^2+5}{2x_0} = \frac{4+5}{2(2)} = \frac{9}{4}$

$$x_1 = 2.25$$

$$x_2 = \frac{x_1^2 + N}{2x_1} = \frac{(2.25)^2+5}{2(2.25)}$$

$$= 2.2361$$

$$x_3 = \frac{(2.2361)^2+5}{2(2.2361)} = 2.2361$$

$\therefore$ The value of $\sqrt{5} = 2.2361$.

2) Find the iterative formula for finding the value of $\frac{1}{N}$, where N is a real no. using Newton Raphson method. Hence evaluate $\frac{1}{26}$ correct to 4 decimal places.

Let $x = \frac{1}{N}$ i.e) $N = \frac{1}{x}$

$F(x) = \frac{1}{x} - N$; $F'(x) = \frac{-1}{x^2}$

Now, $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} = x_n - \frac{\frac{1}{x_n} - N}{-1/x_n^2}$

$$= x_n + x_n(1 - Nx_n) = x_n + x_n - Nx_n^2$$

$x_{n+1} = 2x_n - Nx_n^2$, which is the iterative formula.

To find $\frac{1}{26}$, $N=26$

$$F(x) = \frac{1}{x} - 26 \; ; \; F'(x) = \frac{-1}{x^2}$$

$F(0) = -26$ (-ve)

$F(1) = -25$

$F(2) = -25.5$ (-ve)

Let us take $x_0 = \frac{1}{25} = 0.04$, nearer to given N

Let $x_0 = 0.04$

WKT $x_{n+1} = 2x_n - Nx_n^2$

$x_1 = 2(0.04) - 26(0.04)^2$

$x_1 = 0.0384$

$x_2 = 0.0384$

Since x_1 & x_2 are equal, the value of $\frac{1}{26} = 0.0384$

3) Derive Newton's algorithm for finding the p^{th} root of a number N & find the value of $(24)^{1/3}$

Let $x = N^{1/p}$

$x^p = N$

$\Rightarrow x^p - N = 0$

Let $F(x) = x^p - N$; $F'(x) = px^{p-1}$

WKT, $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} = x_n - \frac{x_n^p - N}{px_n^{p-1}}$

$$x_{n+1} = \frac{Px_n^P - x_n^P + N}{Px_n^{P-1}} = \frac{(P-1)x_n^P + N}{Px_n^{P-1}}$$

To find $(24)^{1/3}$

Here $N = 24$, $P = 3$.

$F(x) = x^P - N$

$F(x) = x^3 - 24$

$F(0) = -24$

$F(1) = -23$

$F(2) = -16$ (-ve)

$F(3) = 3$ (+ve), the root lies between 2 & 3

Since $|F(2)| > |F(3)|$ let us assume $x_0 = 3$

$$x_{n+1} = \frac{(3-1)x_n^3 + 24}{3x_n^{3-1}} = \frac{2x_n^3 + 24}{3x_n^2}$$

$$x_1 = \frac{2x_0^3 + 24}{3x_0^2} = 2.8888$$

$x_2 = 2.8845$

$x_3 = 2.8844$

$x_4 = 2.8844$

Since $x_3 = x_4$, the required root is 2.8844