



## DEPARTMENT OF MATHEMATICS

### Necessary Condition (Cartesian Coordinates):

#### (or) Cauchy - Riemann equations:

If the function  $f(z) = u(x, y) + iv(x, y)$  is analytic in a region  $R$  of the  $z$  plane, then

(i)  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial y}$ ,  $\frac{\partial v}{\partial x}$  and  $\frac{\partial v}{\partial y}$  exists

(ii)  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

at every point in that region.

#### Sufficient Conditions:

If the function  $f(z) = u(x, y) + iv(x, y)$  is analytic in a region  $R$  of the  $z$ -plane if

(i)  $u_x$ ,  $u_y$ ,  $v_x$  &  $v_y$  are exists and all are continuous.

(ii)  $u_x = v_y$  and  $u_y = -v_x$ .

### Necessary Condition (polar Coordinates):

If the function  $w = f(z) = u(r, \theta) + iv(r, \theta)$  is analytic in a region  $R$  of the  $z$ -plane then

(i) if  $\frac{\partial u}{\partial r}$ ,  $\frac{\partial u}{\partial \theta}$ ,  $\frac{\partial v}{\partial r}$  and  $\frac{\partial v}{\partial \theta}$  exists

(ii)  $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$  and  $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

#### Sufficient Conditions:

If the function  $w = f(z) = u(r, \theta) + iv(r, \theta)$  is analytic in a region  $R$  of the  $z$ -plane, then

(i)  $\frac{\partial u}{\partial r}$ ,  $\frac{\partial u}{\partial \theta}$ ,  $\frac{\partial v}{\partial r}$  and  $\frac{\partial v}{\partial \theta}$  exists & all are continuous

(ii)  $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$  and  $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$





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### PROBLEMS:

① Prove that  $w = z^2$  is analytic.

Soln:

We know  $z = x + iy$

$$\therefore w = z^2$$

$$= (x + iy)^2$$

$$= x^2 - y^2 + 2ixy$$

$$u + iv = (x^2 - y^2) + i(2xy)$$

$$u = x^2 - y^2 \quad ; \quad v = 2xy$$

$$u_x = 2x$$

$$v_x = 2y$$

$$u_y = -2y$$

$$v_y = 2x$$

$$u_x = v_y \quad \& \quad u_y = -v_x$$

It satisfies CR equations.

$\Rightarrow w = z^2$  is analytic.

② Determine whether the function  $w = 2xy + i(x^2 - y^2)$  is analytic.

Soln:  $w = 2xy + i(x^2 - y^2)$

$$u + iv = 2xy + i(x^2 - y^2)$$

$$u = 2xy \quad , \quad v = x^2 - y^2$$

$$u_x = 2y \quad , \quad v_x = 2x$$

$$u_y = 2x \quad , \quad v_y = -2y$$

$$u_x \neq v_y$$

$$\& \quad u_y \neq -v_x$$

It doesn't satisfy CR equations

$\Rightarrow w = 2xy + i(x^2 - y^2)$  is not analytic.



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- ③ Verify whether  $f(z) = \sin hz$  is analytic using CR eqns.

Soln:

$$f(z) = \sin hz$$

$$u+iv = \sin h(x+iy)$$

$$= \frac{1}{i} \sin i(x+iy) \quad (\text{multiply \& divide by } i)$$

$$= \frac{1}{i} \sin(ix+iy^2)$$

$$= \frac{1}{i} \sin(ix-y)$$

$$= \frac{1}{i} [\sin ix \cos y - \cos ix \sin y]$$

$$= \frac{1}{i} [i \sin hx \cos y - \cosh x \sin y]$$

$$= \sin hx \cos y - \frac{1}{i} \cosh x \sin y$$

$$= \sin hx \cos y + i \cosh x \sin y \quad \left[ \frac{1}{i} = -i \right]$$

$$u = \sin hx \cos y, \quad v = \cosh x \sin y$$

$$u_x = \cosh x \cos y, \quad v_x = \sinh x \sin y$$

$$u_y = -\sin hx \sin y, \quad v_y = \cosh x \cos y$$

$$u_x = v_y \quad \& \quad u_y = -v_x$$

$\therefore$  It satisfies CR equations.

$\therefore f(z) = \sin hz$  is analytic.

- ④ Show that  $f(z) = |z|^2$  is nowhere analytic.

Soln:

$$f(z) = |z|^2$$

$$u+iv = x^2+y^2$$

$$u = x^2+y^2, \quad v=0$$



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$$u_x = 2x, \quad v_x = 0$$

$$u_y = 2y, \quad v_y = 0$$

$$\Rightarrow u_x \neq v_y \text{ \& } u_y \neq -v_x$$

It doesn't satisfy CR eqns.

$\therefore f(z) = |z|^2$  is nowhere analytic.

⑤ If  $u+iv$  is analytic then  $v-iu$  is also analytic.

Soln:

$u+iv$  is analytic

i.e., CR eqns are satisfied.

$$\text{i.e., } u_x = v_y \text{ \& } u_y = -v_x$$

$$\text{i.e., } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ \& } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

To prove:  $v-iu$  is also analytic.

$$\text{i.e., We have to prove, } \frac{\partial v}{\partial x} = \frac{\partial (-u)}{\partial y} \text{ \& } \frac{\partial v}{\partial y} = -\frac{\partial (-u)}{\partial x}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ \& } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

We know that,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ \& } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ \& } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Hence  $v-iu$  is also analytic.

⑥ If  $w = e^z$ , find  $\frac{dw}{dz}$  using complex variable.

Soln:

$$w = e^z$$

$$u+iv = e^{x+iy} = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$$

$$= e^x \cos y + i e^x \sin y$$

$$u = e^x \cos y, \quad v = e^x \sin y$$

$$u_x = e^x \cos y, \quad v_x = e^x \sin y$$



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[Result: If  $w = f(z) = u + iv$  then

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

finding  $\frac{dw}{dz}$  in terms of partial derivatives w.r.t  $x$ )

$$\Rightarrow \frac{dw}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x (\cos y + i \sin y)$$

$$= e^x e^{iy}$$

$$= e^{x+iy}$$

$$= e^z$$