$$= \frac{\sin 3t}{3}\left[\frac{e^{-t}}{10}(-\cos 3t + 3\sin 3t) - \frac{1}{10}(-1)\right]$$

$$- \frac{\cos 3t}{3}\left[\frac{e^{-t}}{10}(-\sin 3t - 3\cos 3t) - \frac{1}{10}(-3)\right]$$

$$= \left[\frac{e^{-t}}{30}\cdot 3(\sin^2 3t + \cos^2 3t)\right] + \left[\frac{1}{30}(\sin 3t + 3\cos 3t)\right]$$

$$= \frac{e^{-t}}{10} + \frac{1}{30}(\sin 3t + 3\cos 3t)$$

Applications of Laplace transforms to Differential equations:

If $L[f(t)] = F(s)$ then

$L[y'(t)] = s L(y) - y(0)$

$L[y''(t)] = s^2 L(y) - s y(0) - y'(0)$

① Solve the differential equations using LT $y'' + 4y' + 4y = e^{-t}$ given that $y(0) = 0$ and $y'(0) = 0$.

Soln:

$$y'' + 4y' + 4y = e^{-t}$$

Taking LT on both sides,

$$L(y'' + 4y' + 4y) = L(e^{-t})$$

$$L(y'') + 4L(y') + 4L(y) = \frac{1}{s+1}$$

$$[s^2 L(y) - s y(0) - y'(0)] + 4[s L(y) - y(0)] + 4L(y) = \frac{1}{s+1}$$

Given: $y(0) = 0, \quad y'(0) = 0$

$$\Rightarrow [s^2 L(y) - s\times 0 - 0] + 4[s L(y) - 0] + 4L(y) = \frac{1}{s+1}$$

$$\Rightarrow s^2 L(y) + 4sL(y) + 4L(y) = \frac{1}{s+1}$$

$$\Rightarrow (s^2 + 4s + 4)L(y) = \frac{1}{s+1}$$

$$\Rightarrow L(y)(s+2)^2 = \frac{1}{s+1}$$

$$L(y) = \frac{1}{(s+1)(s+2)^2}$$

$$y = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+2)(s+1) + C(s+1)$$

Put $s = -2 \Rightarrow C = -1$

Put $s = -1 \Rightarrow A = 1$

Put $s = 0 \Rightarrow B = -1$

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$y = L^{-1}\left(\frac{1}{(s+1)(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right) - L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= e^{-t} - e^{-2t} - te^{-2t}$$

(2) Solve using LT $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 2e^{-3t}$, $y(0) = 1$, $y'(0) = -2$

Soln:

$$y'' + 6y' + 9y = 2e^{-3t}$$

$$L(y'' + 6y' + 9y) = 2L(e^{-3t})$$

$$L(y'') + 6L(y') + 9L(y) = 2L(e^{-3t})$$

$$\{s^2 L(y) - s\,y(0) - y'(0)\} + 6\{s\,L(y) - y(0)\} + 9L(y) = \frac{2}{s+3}$$

$$\{s^2 L(y) - s + 2\} + 6\{s\,L(y) - 1\} + 9L(y) = \frac{2}{s+3}$$

$$s^2 L(y) + 6s\,L(y) + 9L(y) - (s+4) = \frac{2}{s+3}$$

$$(s^2 + 6s + 9)L(y) = \frac{2}{s+3} + s + 4$$

$$(s+3)^2 L(y) = \frac{2}{s+3} + s + 4$$

$$L(y) = \frac{2}{(s+3)^3} + \frac{s+4}{(s+3)^2}$$

$$y = L^{-1}\left[\frac{2}{(s+3)^3} + \frac{s+4}{(s+3)^2}\right]$$

$$= 2L^{-1}\left(\frac{1}{(s+3)^3}\right) + L^{-1}\left(\frac{s+4}{(s+3)^2}\right)$$

$$= 2L^{-1}\left(\frac{1}{(s+3)^3}\right) + L^{-1}\left[\frac{1}{s+3} + \frac{1}{(s+3)^2}\right]$$

$$= 2e^{-3t}L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s+3}\right) + L^{-1}\left(\frac{1}{(s+3)^2}\right)$$

$$= 2e^{-3t}t^2 + e^{-3t} + te^{-3t}$$