DEPARTMENT OF AEROSPACE ENGINEERING

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UNIT V – FLOW FIELD ANALYSIS AND TURBULENCE MODELS

Two-equation Turbulence Models

3.1 The Modelled " Equation

An exact equation for the dissipation can be derived fromtheNavier-Stokes equation (see, for instance,Wilcox [49]). However, the number of unknown terms is very large and they involve double correlations of fluctuating velocities, and gradients of fluctuating velocities and pressure. It is better to derive an ε equation based on physical reasoning. In the exact equation for ε the production term includes, as in the k equation, turbulent quantities and and velocity gradients. If we choose to include uiuj and ¯Ui,j in the production term and only turbulent quantities in the dissipation term, we take, glancing at the k equation (Eq. 2.38)

$$P_\varepsilon = -c_{\varepsilon 1}\frac{\varepsilon}{k}\left(\bar{U}_{i,j}+\bar{U}_{j,i}\right)\bar{U}_{i,j} \tag{3.1}$$

$$\text{diss.term} = -c_{\varepsilon 2}\rho\frac{\varepsilon^2}{k}.$$

Note that for the production term we have $P_\varepsilon = c_{\varepsilon 1}(\varepsilon/k)P_k$. Now we can write the transport equation for the dissipation as

$$(\rho\bar{U}_j\varepsilon)_{,j} = \left[\left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\varepsilon_{,j}\right]_{,j} + \frac{\varepsilon}{k}\left(c_{\varepsilon 1}P_k - c_{\varepsilon 2}\rho\varepsilon\right) \tag{3.2}$$

For boundary-layer flow Eq. 3.2 reads

$$\frac{\partial\rho\bar{U}\varepsilon}{\partial x}+\frac{\partial\rho\bar{V}\varepsilon}{\partial y} = \frac{\partial}{\partial y}\left[\left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial\varepsilon}{\partial y}\right] + c_{\varepsilon 1}\frac{\varepsilon}{k}\mu_t\left(\frac{\partial\bar{U}}{\partial y}\right)^2 - \rho c_{\varepsilon 2}\frac{\varepsilon^2}{k} \tag{3.3}$$

Wall Functions

The naturalway to treatwall boundaries is tomake the grid sufficiently fine so that the sharp gradients prevailing there are resolved. Often, when computing complex three-dimensional flow, that requires too much computer resources. An alternative is to assume that the flow near the wall behaves like a fully developed turbulent boundary layer and prescribe boundary conditions employing wall functions. The assumption that the flow near the wall has the characteristics of a that in a boundary layer if often not true at all. However, given a maximum number of nodes that we can afford to use in a computation, it is often preferable to use wall functions which allows us to use fine grid in other regions where the gradients of the flow variables are large. In a fully turbulent boundary layer the production term and the dissipation term in the log-law region ($30 < y^+ < 100$) are much larger than the

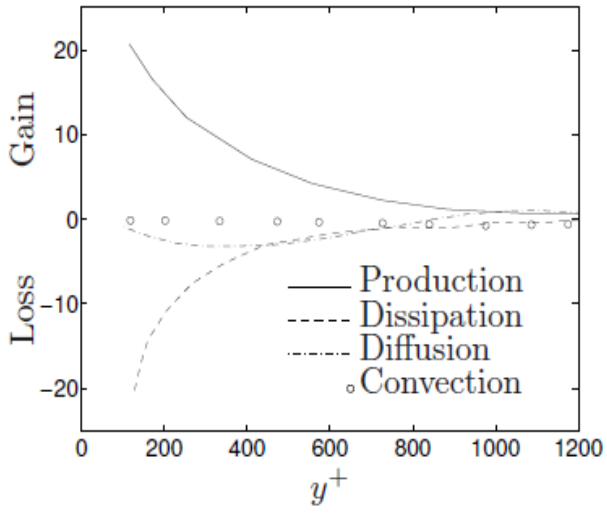

Figure 3.1: Boundary along a flat plate. Energy balance in k equation [46]. $Re_\delta \simeq 4400$, $u_*/U_0 \simeq 0.043$.

other terms, see Fig. 3.1. The log-law we use can be written as

$$\frac{U}{u_*} = \frac{1}{\kappa}\ln\left(\frac{Eu_*y}{\nu}\right) \tag{3.4}$$

$$E = 9.0 \tag{3.5}$$

Comparing this with the standard form of the log-law

$$\frac{U}{u_*} = A\ln\left(\frac{u_*y}{\nu}\right) + B \tag{3.6}$$

we see that

$$\begin{aligned} A &= \frac{1}{\kappa} \\ B &= \frac{1}{\kappa}\ln E. \end{aligned} \tag{3.7}$$

In the log-layer we can write the modelled k equation (see Eq. 2.39) as

$$0 = \mu_t\left(\frac{\partial\bar{U}}{\partial y}\right)^2 - \rho\varepsilon. \tag{3.8}$$

where we have replaced the dissipation term $\rho k^{\frac{3}{2}}/\ell$ by $\rho\varepsilon$. In the log-law region the shear stress $-\rho\overline{uv}$ is equal to the wall shear stress τ_w, see Fig. 2.1. The Boussinesq assumption for the shear stress reads (see Eq. 2.10)

$$\tau_w = -\rho\overline{uv} = \mu_t\frac{\partial\bar{U}}{\partial y} \tag{3.9}$$

Using the definition of the wall shear stress $\tau_w = \rho u_*^2$, and inserting Eq. 3.9 into Eq. 3.8 we get

$$0 = \frac{\overline{uv}^2}{\nu_t} - \varepsilon = \frac{u_\tau^2}{\nu_t} - \varepsilon \tag{3.10}$$

which with $\nu_t = c_\mu \rho k^2/\varepsilon$ gives

$$c_\mu = \left(\frac{u_*^2}{k}\right)^2 \tag{3.11}$$

From experiments we have that in the log-law region of a boundary layer $u_*^2/k \simeq 0.3$ so that $c_\mu = 0.09$.

c_μ constant

When we are using wall functions k and ε are not solved at the nodes adjacent to the walls. Instead they are fixed according to the theory presented above. The turbulent kinetic energy is set from Eq. 3.11, i.e.

b.c. for k

$$k_P = c_\mu^{-1/2} u_*^2 \tag{3.12}$$

where the friction velocity u_* is obtained, iteratively, from the log-law (Eq. 3.4). Index P denotes the first interior node (adjacent to the wall).

The dissipation ε is obtained from observing that production and dissipation are in balance (see Eq. 3.8). The dissipation can thus be written as

b.c. for ε

$$\varepsilon_P = P_k = \frac{u_*^3}{\kappa y} \tag{3.13}$$

where the velocity gradient in the production term $-\overline{uv}\partial U/\partial y$ has been computed from the log-law in Eq. 3.4, i.e.

$$\frac{\partial U}{\partial y} = \frac{u_*}{\kappa y}. \tag{3.14}$$

b.c. for velocity

For the velocity component parallel to the wall the wall shear stress is used as a flux boundary condition (cf. prescribing heat flux in the temperature equation).

When the wall is not parallel to any velocity component, it is more convenient to prescribe the turbulent viscosity. The wall shear stress τ_w is obtained by calculating the viscosity at the node adjacent to the wall from the log-law. The viscosity used in momentum equations is prescribed at the nodes adjacent to the wall (index P) as follows. The shear stress at the wall can be expressed as

$$\tau_w = \mu_{t,P} \frac{\partial \bar{U}}{\partial \eta} \approx \mu_{t,P} \frac{\bar{U}_{\parallel,P}}{\eta}$$

where $\bar{U}_{\parallel,P}$ denotes the velocity parallel to the wall and η is the normal distance to the wall. Using the definition of the friction velocity u_*

$$\tau_w = \rho u_*^2$$

we obtain

$$\mu_{t,P} \frac{U_{\parallel,P}}{\eta} = \rho u_*^2 \rightarrow \mu_{t,P} = \frac{u_*}{U_{\parallel,P}} \rho u_* \eta$$

Substituting $u_*/\bar{U}_{\parallel,P}$ with the log-law (Eq. 3.4) we finally can write

$$\mu_{t,P} = \frac{\rho u_* \eta \kappa}{\ln(E\eta^+)}$$

where $\eta^+ = u_*\eta/\nu$.

3.3 The $k-\varepsilon$ Model

In the $k-\varepsilon$ model the modelled transport equations for k and ε (Eqs. 2.38, 3.2) are solved. The turbulent length scale is obtained from (see Eq. 1.12,2.37)

$$\ell = \frac{k^{3/2}}{\varepsilon}. \tag{3.15}$$

The turbulent viscosity is computed from (see Eqs. 2.11, 2.36, 1.12)

$$\nu_t = c_\mu k^{1/2} \ell = c_\mu \frac{k^2}{\varepsilon}. \tag{3.16}$$

We have five unknown constants $c_\mu, c_{\varepsilon 1}, c_{\varepsilon 2}, \sigma_k$ and σ_ε, which we hope should be universal i.e same for all types of flows. Simple flows are chosen where the equation can be simplified and where experimental data are used to determine the constants. The c_μ constant was determined above (Subsection 3.2). The k equation in the logarithmic part of a turbulent boundary layer was studied where the convection and the diffusion term could be neglected.

$c_{\varepsilon 1}$ constant

In a similar way we can find a value for the $c_{\varepsilon 1}$ constant . We look at the ε equation for the logarithmic part of a turbulent boundary layer, where the convection term is negligible, and utilizing that production and dissipation are in balance $P_k = \rho\varepsilon$, we can write Eq. 3.3 as

$$0 = \underbrace{\frac{\partial}{\partial y}\left[\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right]}_{D_\varepsilon} + (c_{\varepsilon 1} - c_{\varepsilon 2})\,\rho\frac{\varepsilon^2}{k} \tag{3.17}$$

The dissipation and production term can be estimated as (see Sub-section 3.2)

$$\begin{aligned} \varepsilon &= \frac{k^{3/2}}{\ell} \\ P_k &= \rho\frac{u_*^3}{\kappa y}, \end{aligned} \tag{3.18}$$

which together with $P_k = \rho\varepsilon$ gives

$$\ell = \kappa c_\mu^{-3/4} y. \tag{3.19}$$

In the logarithmic layer we have that $\partial k/\partial y = 0$, but from Eqs. 3.18, 3.19 we find that $\partial\varepsilon/\partial y \neq 0$. Instead the diffusion term in Eq. 3.17 can be rewritten using Eqs. 3.18, 3.19, 3.16 as

$$D_\varepsilon = \frac{\partial}{\partial y}\left[\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial}{\partial y}\left(\frac{k^{3/2}}{\kappa c_\mu^{-3/4} y}\right)\right] = \frac{k^2\kappa^2}{\sigma_\varepsilon \ell^2 c_\mu^{1/2}} \tag{3.20}$$

Inserting Eq. 3.20 and Eq. 3.18 into Eq. 3.17 gives

$$c_{\varepsilon 1} = c_{\varepsilon 2} - \frac{\kappa^2}{c_\mu^{1/2}\sigma_\varepsilon} \tag{3.21}$$

$c_{\varepsilon 2}$ constant

The flow behind a turbulence generating grid is a simple flow which allows us to determine the $c_{\varepsilon 2}$ constant. Far behind the grid the velocity gradients are very small which means that $P_k = 0$. Furthermore $V = 0$ and the diffusion terms are negligible so that the modelled k and ε equations (Eqs. 2.38, 3.2) read

$$\rho\bar{U}\frac{dk}{dx} = -\rho\varepsilon \tag{3.22}$$

$$\rho\bar{U}\frac{d\varepsilon}{dx} = -c_{\varepsilon 2}\rho\frac{\varepsilon^2}{k} \tag{3.23}$$

Assuming that the decay of k is exponential $k \propto x^{-m}$, Eq. 3.22 gives $\varepsilon \propto -mx^{-m-1}$. Insert this in Eq. 3.22, derivate to find $d\varepsilon/dx$ and insert it into Eq. 3.23 yields

$$c_{\varepsilon 2} = \frac{m+1}{m} \tag{3.24}$$

Experimental data give $m = 1.25 \pm 0.06$ [46], and $c_{\varepsilon 2} = 1.92$ is chosen.

We have found three relations (Eqs. 3.11, 3.21, 3.24) to determine three of the five unknown constants. The last two constants, σ_k and σ_ε, are optimized by applying the model to various fundamental flows such as flow in channel, pipes, jets, wakes, etc. The five constants are given the following values: $c_\mu = 0.09$, $c_{\varepsilon 1} = 1.44$, $c_{\varepsilon 2} = 1.92$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.31$.

3.4 The $k-\omega$ Model

The $k-\omega$ model is gaining in popularity. The model was proposed by Wilcox [48, 49, 39]. In this model the standard k equation is solved, but as a length determining equation ω is used. This quantity is often called *specific dissipation* from its definition $\omega \propto \varepsilon/k$. The modelled k and ω equation read

$$(\rho\bar{U}_j k)_{,j} = \left[\left(\mu + \frac{\mu_t}{\sigma_k^\omega}\right)k_{,j}\right]_{,j} + P_k - \beta^*\omega k \tag{3.25}$$

$$
\begin{aligned}
(\rho\bar{U}_j\omega)_{,j} &= \left[\left(\mu+\frac{\mu_t}{\sigma_\omega}\right)\omega_{,j}\right]_{,j}+\frac{\omega}{k}\left(c_{\omega 1}P_k-c_{\omega 2}\rho k\omega\right) \qquad (3.26)\\
\mu_t &= \rho\frac{k}{\omega},\ \varepsilon=\beta^*\omega k.
\end{aligned}
$$

The constants are determined as in Sub-section 3.3: $\beta^*=0.09$, $c_{\omega 1}=5/9$, $c_{\omega 2}=3/40$, $\sigma_k^\omega=2$ and $\sigma_\omega=2$.

When wall functions are used k and ω are prescribed as (cf. Sub-section 3.2):

$$
k_{wall}=(\beta^*)^{-1/2}u_*^2,\ \omega_{wall}=(\beta^*)^{-1/2}\frac{u_*}{\kappa y}. \qquad (3.27)
$$

In regions of low turbulence when both k and ε go to zero, large numerical problems for the $k-\varepsilon$ model appear in the ε equation as k becomes zero. The destruction term in the ε equation includes ε^2/k, and this causes problems as $k\to 0$ even if ε also goes to zero; they must both go to zero at a correct rate to avoid problems, and this is often not the case. On the contrary, no such problems appear in the ω equation. If $k\to 0$ in the ω equation in Eq. 3.25, the turbulent diffusion term simply goes to zero. Note that the production term in the ω equation does not include k since

$$
\frac{\omega}{k}c_{\omega 1}P_k=\frac{\omega}{k}c_{\omega 1}\mu_t\left(\frac{\partial\bar{U}_i}{\partial x_j}+\frac{\partial\bar{U}_j}{\partial x_i}\right)\frac{\partial\bar{U}_i}{\partial x_j}=c_{\omega 1}\beta^*\left(\frac{\partial\bar{U}_i}{\partial x_j}+\frac{\partial\bar{U}_j}{\partial x_i}\right)\frac{\partial\bar{U}_i}{\partial x_j}.
$$

The standard $k-\omega$ model can – contrary to the standard $k-\varepsilon$ model – be used as a low-Re number model all the way to the wall (including the viscous sublayer). In that case, the wall boundary condition for k is simply $k=0$ and ω is fixed at the wall-adjacent cells according to Eq. 4.26.

In Ref. [38] the $k-\omega$ model was used to predict transitional, recirculating flow.

3.5 The $k-\tau$ Model

One of the most recent proposals is the $k-\tau$ model of Speziale *et al.* [42] where the transport equation for the turbulent time scale τ is derived. The exact equation for $\tau=k/\varepsilon$ is derived from the exact k and ε equations. The modelled k and τ equations read

$$
(\rho\bar{U}_jk)_{,j}=\left[\left(\mu+\frac{\mu_t}{\sigma_k^\tau}\right)k_{,j}\right]_{,j}+P_k-\rho\frac{k}{\tau} \qquad (3.28)
$$

$$
\begin{aligned}
(\rho\bar{U}_j\tau)_{,j} &= \left[\left(\mu+\frac{\mu_t}{\sigma_{\tau 2}}\right)\tau_{,j}\right]_{,j}+\frac{\tau}{k}\left[(1-c_{\varepsilon 1})P_k+(c_{\varepsilon 2}-1)\frac{k}{\tau}\right] \quad (3.29)\\
&+ \frac{2}{k}\left(\mu+\frac{\mu_t}{\sigma_{\tau 1}}\right)k_{,j}\tau_{,j}-\frac{2}{\tau}\left(\mu+\frac{\mu_t}{\sigma_{\tau 2}}\right)\tau_{,j}\tau_{,j}\\
\mu_t &= c_\mu\rho k\tau,\ \varepsilon=k/\tau
\end{aligned}
$$

The constants are: c_μ, $c_{\varepsilon 1}$ and $c_{\varepsilon 2}$ are taken from the $k-\varepsilon$ model, and $\sigma_k^\tau = \sigma_{\tau 1}=\sigma_{\tau 2}=1.36$.