Lagrange's Linear Equations

The equations of the form $Pp+Qq=R$ is known as Lagrange's Linear equation, where P, Q are functions of x, y, z

To solve this equation, it is enough to solve the subsidary equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

The auxillary eqn can be solved in 2 ways

1. Method of grouping
2. Method of Multipliers

Method of Grouping:-

$$Pp+Qq=R$$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

I $\quad \int\frac{dx}{P}=\int\frac{dy}{Q}$

II $\quad \int\frac{dy}{Q}=\int\frac{dz}{R}$

III $\quad \int\frac{dz}{R}=\int\frac{dx}{P}$

$\Rightarrow \phi(c_1,c_2)=0.$

1. Solve $x^2p+y^2q=z^2$

$$Pp+Qq=R$$

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{z^2}.$$

$$\Rightarrow \int\frac{dx}{P}=\int\frac{dy}{Q}$$

$$\int\frac{dx}{x^2}=\int\frac{dy}{y^2}$$

$$\int x^{-2}dx=\int y^{-2}dy$$

$$-x^{-1}+c=-y^{-1}+c$$

$$\frac{-1}{x}+\frac{1}{y}=c_1 \rightarrow (1)$$

$$\Rightarrow \int\frac{dy}{Q}=\int\frac{dz}{R}$$

$$\int\frac{dy}{y^2}=\int\frac{dz}{z^2}$$

$$\int y^{-2}dy=\int z^{-2}dz$$

$$\frac{y^{-1}}{-1}+c=\frac{z^{-1}}{-1}+c$$

$$\frac{-1}{y}+\frac{1}{z}=c_2 \rightarrow (2)$$

Combining eqns (1) & (2)

$$\phi(c_1,c_2)=0.$$

$$\phi\left(\frac{-1}{x}+\frac{1}{y},\ \frac{-1}{y}+\frac{1}{z}\right)=0.$$

2) $\frac{y^2 z}{x} p + xzq = y^2$

$Pp + Qq = R$

$$\frac{dx}{P} = \frac{dy}{Q}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}.$$

$$\text{I} \Rightarrow \int \frac{dx}{\frac{y^2 z}{x}} = \int \frac{dy}{xz}$$

$$\int \frac{x\,dx}{y^2 z} = \int \frac{dy}{xz}$$

$$\int x^2 dx = \int y^2 dy$$

$$\frac{x^3}{3} + c = \frac{y^3}{3} + c$$

$$\Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = c_1 \rightarrow (1)$$

$$\text{III} \Rightarrow \int \frac{dz}{y^2} = \int \frac{dx}{\frac{y^2 z}{x}}$$

$$\int \frac{dz}{y^2} = \int \frac{x\,dx}{y^2 z}$$

$$\int z\,dz = \int x\,dx$$

$$\frac{z^2}{2} + c = \frac{x^2}{2} + c$$

$$\frac{z^2}{2} - \frac{x^2}{2} = c_2 \rightarrow (2)$$

$$\phi\left(\frac{x^3}{3} - \frac{y^3}{3}, \frac{z^2}{2} - \frac{x^2}{2}\right) = 0.$$