Type-II Clairaut's form $z = px + qy + f(p,q)$

Working Rule:

Complete Integral : Replace $p \to a$ and $q \to b$

Singular Integral : $\frac{\partial z}{\partial a} = 0$ and $\frac{\partial z}{\partial b} = 0$

General Integral : put $b = \phi(a)$ in complete Integral.

1. Solve: $z = px + qy + pq$.

Soln:- Given $z = px + qy + pq \to (1)$

Complete Integral :

$z = ax + by + ab \to (2)$ [Replace $p \to a$, $q \to b$]

Singular Integral :

$\frac{\partial z}{\partial a} = 0$ and $\frac{\partial z}{\partial b} = 0$

$x + b = 0$ $y + a = 0$

$b = -x$ $a = -y$

Sub 'a' and 'b' in (2)

$z = -yx - xy - y(-x) \Rightarrow z = -xy$

General Integral :

Sub $b = \phi(a)$ in (2)

$z = ax + \phi(a)y + a\phi(a) \to (3)$

diff p wrt 'a', $\frac{\partial z}{\partial a} = 0$

$\Rightarrow x + \phi'(a)y + a\phi'(a) + \phi(a) = 0 \to (4)$

Eliminate 'a' b/w (3) & (4) we get the general soln.

2. Solve: $z = px + qy + p^2 - q^2$

Soln Given $z = px + qy + p^2 - q^2 \to (1)$

Complete Integral :

$z = ax + by + a^2 - b^2 \to (2)$ [Replace $p \to a$ & $q \to b$]

Singular Integral :

$\frac{\partial z}{\partial a} = 0$	$\frac{\partial z}{\partial b} = 0$
$x + 2a = 0$	$y - 2b = 0$
$2a = -x$	$y = 2b$
$a = -x/2$	$b = y/2$

Sub a & b in ②

$$z = \frac{-x}{2}x + \frac{y}{2}y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$$

$$= \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$4z = -2x^2 + 2y^2 + x^2 - y^2$$

$$= -x^2 + y^2$$

$$\Rightarrow 4z = y^2 - x^2$$

General Integral:

Sub $b = \phi(a)$ in ②

$$z = ax + \phi(a)y + a^2 - (\phi(a))^2 \rightarrow ③$$

diff p wrt 'a', $\Rightarrow \frac{\partial z}{\partial a} = 0$

$$\frac{\partial z}{\partial a} \Rightarrow x + \phi'(a)y + 2a - 2\phi(a)\phi'(a) = 0 \rightarrow ④$$

Eliminate 'a' b/w ③ & ④ we get the general soln.