UNIT–III,TWODIMENSIONALRANDOMVARIABLES

PART-A

1. Thebivariate random variableXandYhasthepdf

   f(x,y)= $f(x,y)=\begin{cases} kx^2(8-y), x<y<2x \\ 0\le x<2 \end{cases}$ find k.

   Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx=1$$

$$\int_{0}^{2}\int_{x}^{2x} kx^2(8-y)\,dy\,dx=1 \qquad k\int_0^2 x^2\left(8y-\frac{y^2}{2}\right)_x^{2x}dx=1$$

$$k\int_0^2 x^2\left(16x-\frac{4x^2}{2}-8x+\frac{x^2}{2}\right)dx=1 \qquad k\int_0^2\left(16x^3-2x^4-8x^3+\frac{x^4}{2}\right)dx=1$$

$$k\int_0^2\left(8x^3-\frac{3x^4}{2}\right)dx=1 \qquad k\left[\frac{8x^4}{4}-\frac{3x^5}{10}\right]_0^2=1$$

$$k\left[32-\frac{48}{5}\right]=1 \qquad k\left[\frac{112}{5}\right]=1$$

$$k\left[\frac{112}{5}\right]=1$$

$$k=\frac{5}{112}$$

2. The joint pdf of random variable x and y is given by $f(x,y)=kxye^{-(x^2+y^2)}, x>0, y>0$ find the value of k.

   Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1$$

$$\int_0^\infty\int_0^\infty kxye^{-(x^2+y^2)}dydx=1$$

$$k\int_0^\infty ye^{-y^2}dy\int_0^\infty xe^{-x^2}dx=1 \qquad \left[\int_0^\infty xe^{-x^2}dx=\frac{1}{2}\right]$$

$$k\frac{1}{2}\frac{1}{2}=1, k=4$$

3. IfX andYhavejointpdf$f(x,y)=\begin{cases} x+y, 0<x<1, 0<y<1 \\ 0, \quad \text{otherwise} \end{cases}$.checkwhetherXand Y are independent.

Ans:

ThemarginaldensityofXis

$$f(x)=\int_{-\infty}^{\infty}f(x,y)dy \qquad f(x)=\int_0^1(x+y)dy$$

$$f(x)=\left[xy+\frac{y^2}{2}\right]_0^1 \qquad f(x)=x+\frac{1}{2}$$

$$f(y)=\int_{-\infty}^{\infty}f(x,y)dy \qquad f(y)=\int_0^1(x+y)dy$$

$$f(y)=\left[\frac{x^2}{2}+xy\right]_0^1 \qquad f(y)=\frac{1}{2}+y$$

$$f(x).f(y)=\left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \qquad f(x).f(y)\neq f(x,y)$$

4. LetXandYhavej.d.ff(x,y)=2,0<x<y<1.Findm.d.f Ans:

MarginaldensityofX isgivenby

$$f(x)=\int_{-\infty}^{\infty} f(x,y)dy \qquad =\int_{x}^{1} 2dy$$

$$=2[y]_x^1$$

$$=2(1-x), 0<x<1.$$

MarginaldensityfunctionofYisgiven by

$$f(y)=\int_{-\infty}^{\infty} f(x,y)dy \qquad =\int_{0}^{y} 2dx \qquad =2[x]_0^y \qquad =2y, 0<y<1.$$

5. Thej.d.foftherandomvariablesXandYisgivenby $f(x,y)=\begin{cases} 8xy, 0<x<1, 0<y<x \\ 0, \quad \text{otherwise} \end{cases}$ .findf(x).

Ans:

$$f_x(x)=\int_{-\infty}^{\infty} f(x,y)dy$$

$$=\int_{0}^{x} 8xydy \qquad =8x\left(\frac{y^2}{2}\right)_0^x$$

$$=8x\left(\frac{x^2}{2}\right)$$

$$f_x(x)=4x^3, 0<x<1$$

6. Given $f(x,y)=\begin{cases} cx(x-y), 0<x<2, -x<y<x \\ 0, \quad \text{otherwise} \end{cases}$,find c.

Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx=1$$

$$\int_0^2\int_{-x}^{2x} cx(x-y)\,dy\,dx=1$$

$$c\int_0^2\left[x^2y-x.\frac{y^2}{2}\right]_{-x}^{2x}dx=1$$

$$c\int_0^2\left[x^3-\frac{x^2}{2}+x^3+\frac{x^3}{2}\right]dx=1$$

$$c\int_0^2 2x^3dx=1$$

$$2c\left[\frac{x^4}{4}\right]_0^2=1 \qquad 2c\left[\frac{16}{4}\right]=1 \qquad c=\frac{1}{8}$$

6. Thejointp.d.fofabivariaterandomvariable(X,Y)isgiven by

$$f(x,y)=\begin{cases}kxy, 0<x<1, 0<y<1\\ 0, \quad otherwise\end{cases}$$,findK.

Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1$$

$$\int_0^1\int_0^1 kxy\,dx\,dy=1 \qquad k\int_0^1\left[\frac{x^2}{2}y\right]_0^1dy=1$$

$$k\int_0^1\frac{y}{2}dy=1 \qquad k\left[\frac{y^2}{4}\right]_0^1=1$$

$$k=4$$

7. Ifthejoint pdfof(x,y)is $f(x,y)=\frac{1}{4}, 0<x,y<1$,find $p(x+y\le 1)$.

Ans:

$$p(x+y\le 1)=p(x\le 1-y)$$

$$=\int_0^1\int_0^{1-y} f(x,y)\,dx\,dy$$

$$=\int_0^1\int_0^{1-y}\frac{1}{4}\,dx\,dy \qquad =\frac{1}{4}\int_0^1[x]_0^{1-y}\,dy$$

$$=\frac{1}{4}\int_0^1[(1-y)]\,dy \qquad =\frac{1}{4}\left[y-\frac{y^2}{2}\right]_0^1$$

$$=\frac{1}{4}\left[1-\frac{1}{2}\right] \qquad =\frac{1}{4}\cdot\frac{1}{2} \qquad =\frac{1}{8}$$

8. Two random variables X and Y have joint pdf $f(x,y)=\begin{cases}\frac{xy}{96}, & 0<x<4, 1<y<5\\ 0, & \text{otherwise}\end{cases}$, find E(x).

Ans:

$$E(x)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy$$

$$=\int_1^5\int_0^4 x\left(\frac{xy}{96}\right)dx\,dy \qquad =\frac{1}{96}\int_1^5\left[y\cdot\frac{x^3}{3}\right]_0^4 dy$$

$$=\frac{1}{96}\int_1^5\frac{64}{3}y\,dy \qquad =\frac{64}{288}\left[\frac{y^2}{2}\right]_1^5$$

$$=\frac{2}{9}\left[\frac{25}{2}-\frac{1}{2}\right] \qquad =\frac{1}{9}(24)$$

$$=\frac{8}{3}.$$

9. LetXbeaRandomvariablewithpdf $f(x)=\frac{1}{2}, -1\leq x\leq 1$, andlet $Y=X^2$, findE(Y).

Ans:

$$Y=x^2$$

$$E(Y)=E(x^2)$$

$$\int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-\infty}^{\infty} x^2\left(\frac{1}{2}\right)dx = \frac{1}{2}\left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{1}{6}(2)=\frac{1}{3}.$$

10. Ifthejointpdfof(x,y)isgivenby $f(x,y)=x+y, 0\leq x,y\leq 1$. Find $E(XY)$.

Ans:

$$E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)dxdy$$

$$=\int_0^1\int_0^1 xy(x+y)dxdy$$

$$=\int_0^1\int_0^1 \left(x^2y+xy^2\right)dxdy$$

$$=\int_0^1\left(\frac{x^3}{3}y+\frac{x^2}{2}y^2\right)_0^1 dy$$

$$=\int_0^1\left(\frac{y}{3}+\frac{y^2}{2}\right)dy$$

$$\left(\frac{y^2}{6}+\frac{y^3}{6}\right)_0^1=\frac{2}{6}=\frac{1}{3}$$

11. Findtheacuteanglebetweenthetwolinesofregression Ans:

Theacuteanglebetweenthetwolinesofregression is $\tan\theta=\frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\right)$.

12. Statetheequationof thetwo regressionlines.Whatistheformulaforcorrelationcoefficient Ans:

XonYis $(x-\bar{x})=b_{xy}(y-\bar{y})$ andYonXis $(y-\bar{y})=b_{yx}(x-\bar{x})$.

Correlationcoefficient $r = \sqrt{b_{xy}.b_{yx}}$.

13. IfXandYareindependentrandomvariableswithvariance2and 3.Findthevarianceof 3X+4Y. Ans:

$$Var(x)=2, Var(y)=3$$

$$Var(3X+4Y)=3^2 Var(X)+4^2 Var(Y)$$

$$=9Var(X)+16Var(Y)$$

$$=9*2+16*3$$

$$=66$$

14. Thejointpdf of(X,Y) isgivenby$e^{-(x+y)}, 0<x,y<\infty$. AreXandYindependent? Ans :

$$f(x)=\int_{-\infty}^{\infty} f(x,y)dy$$

$$=\int_0^{\infty} e^{-x}e^{-y}dy$$

$$=e^{-x}\left(-e^{-y}\right)_0^{\infty}=-e^{-x}(0-1)=e^{-x}.$$

$$f(y)=\int_{-\infty}^{\infty} f(x,y)dx$$

$$=\int_0^{\infty} e^{-x}e^{-y}dx$$

$$=e^{-y}\left(-e^{-x}\right)_0^{\infty}=-e^{-y}(0-1)=e^{-y}.$$

$$f(x)f(y)=e^{-x}e^{-y}=e^{-(x+y)}=f(x,y)$$ Therefore,XanyYareindependent.

15. The two lines of regression are$8x-10y+66 = 0, 40x-18y-214 = 0$. Find the mean value of X and Y.

Ans:

$8x-10y=-66$____________(1)

$40x-18y=214$___________(2)

Solving(1)and(2),weget $x=104, y=17$

Mean of X = 13

MeanofY=17.

16. Thetworegressionlinesare $x=\frac{9}{20}y+\frac{107}{20}$, $y=\frac{4}{5}x+\frac{33}{5}$. Findcorrelationcoefficient?

Ans:

$$r=\sqrt{b_{xy}\cdot b_{yx}}$$

Here, $b_{xy}=\frac{9}{20}$, $b_{yx}=\frac{4}{5}$

$$r=\sqrt{\frac{9}{20}\times\frac{4}{5}}=0.6$$

17. IfthepdfofXis $f(x)=2x, 0<x<1$. Findthe pdf of $y=3x+1$.

Ans:

Given $y=3x+1$

$$\frac{dy}{dx}=3\Rightarrow\frac{dx}{dy}=\frac{1}{3}$$

$$f_Y(y)=f(x)\left|\frac{dx}{dy}\right| \qquad Y=3X+1,$$

$$=2x\left(\frac{1}{3}\right) \qquad 3X=Y-1$$

$$=2\left(\frac{y-1}{3}\right)\left(\frac{1}{3}\right) \qquad X=\frac{Y-1}{3}$$

$$=\frac{2}{9}(y-1).$$

Whenx=0,y=1

Whenx=1,y=4 $\qquad f(y)=\frac{2}{9}(y-1), 1<y<4$

18. StateCentralLimittheorem.

Liapounoff's form:

If $X_i(1,2,....,n)$ be n independentrandomvariablesuchthat $E(X)=\mu$ and $Var(X)=\sigma^2$, thenundercertaingeneralconditions,therandomvariable $S_n=X_1+X_2+\ldots+X_n$ is asymptotically normal with mean μ and standard deviation σ.

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