TAYLOR SERIES METHOD:

Consider the first order differential eqn.

$$\frac{dy}{dx} = f(x,y) \quad \text{with } y(x_0) = y_0.$$

Hence the Taylor's series expansion of $y(x)$ is given by

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots$$

Let $x_1 = x_0 + h$

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \cdots$$

Now let $x_2 = x_1 + h$

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \cdots$$

(1) Using Taylor Series method find y at $x = 0.1$ if $\frac{dy}{dx} = x^2 y - 1$, $y(0) = 1$

Soln:

Gn: $y' = x^2y - 1$

$x_0 = 0, \quad y_0 = 1, \quad h = 0.1$

Taylor series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \cdots$$

$y' = x^2y - 1 \quad \Rightarrow y_0' = -1$

$y'' = 2xy + x^2y' \quad \Rightarrow y_0'' = 0$

$y''' = 2xy' + 2y + 2xy' + x^2y'' \Rightarrow y_0''' = 2$

$= 2y + 4xy' + x^2y''$

$y^{iv} = 2y' + 4xy'' + 4y' + x^2y''' + 2xy'' \quad \Rightarrow y_0^{iv} = -6$

$= 6y' + 6xy'' + x^2y'''$

$$\text{Now } y_1 = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(-6) + \cdots$$

$= 1 - 0.1 + 0.00033 - 0.000025$

$= 0.900305$

Alternate Method :

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{iv} + \cdots$$

$$= 1 + (x-0)(-1) + \frac{(x)^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots$$

$$= 1 - x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{x^4}{4!}(-6) + \cdots$$

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^2}{2!} + 2\frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!}(-6) + \cdots$$

$$= 0.900305$$

2) Solve $y' = x + y$; $y(0) = 1$ by Taylor's series method. Find the values y at $x = 0.1$ and $x = 0.2$.

Soln:

$y' = x + y$

$x_0 = 0$; $y_0 = 1$ $h = 0.1$

Taylor series is

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$y' = x + y \Rightarrow y_0' = 1$

$y'' = 1 + y' \Rightarrow y'' = 2$

$y''' = y'' \Rightarrow y''' = 2$

$y^{iv} = y''' \Rightarrow y^{iv} = 2$

$$y = 1 + x(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(2) + \cdots$$

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \cdots$$

$$y(0.1) = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12} + \cdots$$

$$= 1 + 0.1 + 0.01 + 0.00033 + 0.00000833$$

$$= 1.1103$$

$$y(0.2) = 1 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{12} + \cdots$$

$$= 1 + 0.2 + 0.04 + 0.00267 + 0.00013$$

$$= 1.2428.$$

1) Using Taylor method, compute $y(0.2)$ & $y(0.4)$ correct to 4 decimal places y_n. $y' = 1 - 2xy$ and $y(0) = 0$.

Soln: $0.2 \rightarrow 0.194752003$

$0.4 \rightarrow 0.359883723$

TAYLOR SERIES METHOD FOR SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

Consider the eqn. of the type $\frac{dy}{dx} = f_1(x, y, z)$, $\frac{dz}{dx} = f_2(x, y, z)$ with initial conditions $y(x_0) = y_0$, $z(x_0) = z_0$ can be solved by Taylor series method.

Solve the system of equations $\frac{dy}{dx} = z - x^2$, $\frac{dz}{dx} = y + x$ with $y(0) = 1$, $z(0) = 1$ by taking $h = 0.1$, to get $y(0.1)$ and $z(0.1)$. Here y and z are dependent variables and x is independent.

Soln: Here $x_0 = 0$, $y_0 = 1$, $z_0 = 1$

$y' = z - x^2 \Rightarrow y_0' = z_0 - x_0^2 = 1 \quad z' = x + y \Rightarrow z_0' = x_0 + y_0 = 1$

$y'' = z' - 2x \Rightarrow y_0'' = z_0' - 2x_0 = 1 \quad z'' = 1 + y' \Rightarrow z_0'' = 1 + y_0' = 2$

$y''' = z'' - 2 \Rightarrow y_0''' = z_0'' - 2 = 0 \quad z''' = y'' \Rightarrow z_0''' = y_0'' = 1$

$y^{iv} = z''' \Rightarrow y_0^{iv} = z_0''' = 1 \quad z^{iv} = y''' \Rightarrow z_0^{iv} = y_0''' = 0$

By Taylor Series for y_1 and z_1 we have,

$$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$$

$$= 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(1) + \dots$$

$$= 1.1050$$

$$z_1 = z(0.1) = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \dots$$

$$= 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(0) + \dots$$

$$= 1.1101$$

2) Find $y(0.3)$ & $z(0.3)$ given $\frac{dz}{dx} = -xy$, $\frac{dy}{dx} = 1 + xz$ with $y(0) = 0$ & $z(0) = 1$

Soln Here $x_0 = 0$, $y_0 = 0$, $z_0 = 1$ & $h = 0.3$

By Taylor Series for y_1 and z_1 we have.

$$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

$$= 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(1) + \dots$$

$$= 1.1050$$

$$z_1 = z(0.1) = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \dots$$

$$= 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(0) + \dots$$

$$= 1.1101$$

2) Find $y(0.3)$ & $z(0.3)$ given $\frac{dz}{dx} = -xy$, $\frac{dy}{dx} = 1 + xz$ with $y(0) = 0$ & $z(0) = 1$

Soln. Here $x_0 = 0$, $y_0 = 0$, $z_0 = 1$ & $h = 0.3$