Puzzle: The Disappearing Population

A small island has a population of bacteria that follows the logistic growth model:

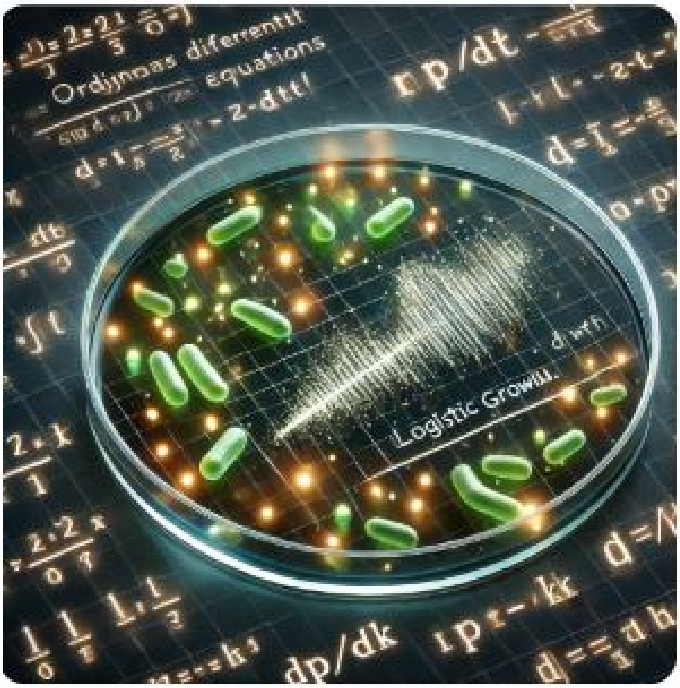
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

where:

- $P(t)$ is the population at time t,
- r is the intrinsic growth rate,
- K is the carrying capacity of the environment.

Initially, there were 100 bacteria, and the carrying capacity of the island is 1000 bacteria. Suppose the intrinsic growth rate is **0.5 per hour**.

◆ **Question:** How long does it take for the population to reach **500 bacteria**?



Let's solve the logistic growth puzzle step by step.

Given Logistic Growth Model:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

where:

- $r = 0.5$ per hour (growth rate)
- $K = 1000$ (carrying capacity)
- $P(0) = 100$ (initial population)
- We need to find the time t when $P(t) = 500$.

Step 1: Solution of the Logistic Equation

The standard solution to the logistic equation is:

$$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right) e^{-rt}}$$

Substituting the given values:

$$P(t) = \frac{1000}{1 + \left(\frac{1000-100}{100}\right) e^{-0.5t}}$$

$$P(t) = \frac{1000}{1 + 9e^{-0.5t}}$$

Step 2: Solve for t when $P(t) = 500$

$$500 = \frac{1000}{1 + 9e^{-0.5t}}$$

Multiply both sides by $1 + 9e^{-0.5t}$:

$$500(1 + 9e^{-0.5t}) = 1000$$

$$1 + 9e^{-0.5t} = 2$$

$$9e^{-0.5t} = 1$$

Divide by 9:

$$e^{-0.5t} = \frac{1}{9}$$

Take the natural logarithm on both sides:

$$-0.5t = \ln\left(\frac{1}{9}\right)$$

Since $\ln(1/9) = -\ln 9$, we get:

$$-0.5t = -\ln 9$$

$$t = \frac{\ln 9}{0.5}$$

$$t = 2\ln 3$$

Final Answer

Approximating $\ln 3 \approx 1.0986$, we get:

$$t \approx 2 \times 1.0986 = 2.197 \text{ hours}$$

So, it takes **about 2.2 hours** for the bacteria population to reach **500**.