Unit III – Dynamic Programming

- Dynamic Programming
  - Computing a Binomial Coefficient
  - Warshall's algorithm
  - **Floyd's algorithm**
  - **Optimal Binary Search Trees**
  - Knapsack Problem and Memory functions

Floyd's algorithm

- Weighted connected graph – all pair shortest path
- Algorithm

```
ALGORITHM  Floyd(W[1..n, 1..n])
    //Implements Floyd's algorithm for the all-pairs shortest-paths problem
    //Input: The weight matrix W of a graph with no negative-length cycle
    //Output: The distance matrix of the shortest paths' lengths
    D ← W  //is not necessary if W can be overwritten
    for k ← 1 to n do
        for i ← 1 to n do
            for j ← 1 to n do
                D[i, j] ← min{D[i, j], D[i, k] + D[k, j]}
    return D
```

- Time Complexity – $O(n^3)$

Optimal Binary Search Tree

Cost Matrix

C[i ,i-1] = 0

C[i, i] = 0

C[i, j] = formula

Root Matrix

R[i, i] = i

R[i, j] = k (min)

	0	1	2	3	4
1	**0**	**0.1**	0.4	1.1	1.7
2		**0**	**0.2**	0.8	1.4
3			**0**	**0.4**	1.0
4				**0**	**0.3**
5					**0**

	0	1	2	3	4
1		**1**	2	3	3
2			**2**	3	3
3				**3**	3
4					**4**
5					

CT

	0	1	2	3	4
1	0	0.1	0.4	1.1	1.7
2		0	0.2	0.8	1.4
3			0	0.4	1.0
4				0	0.3
5					0

RT

	0	1	2	3	4
1		1	2	3	3
2			2	3	3
3				3	3
4					4
5					

$C[i, i-1] = 0$

$C[i, i] = P_i$

$C[i, j]$ = Formula

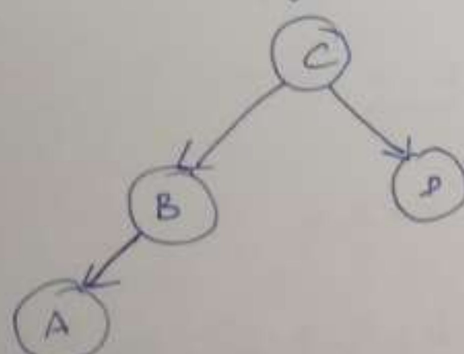
$R[i, i] = i$

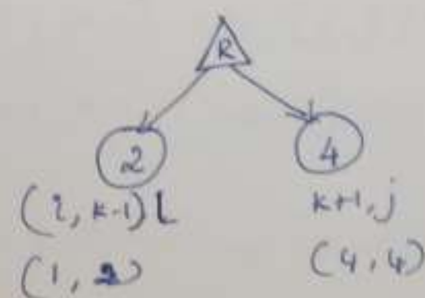
$R[i, j] = k$ (minimum)

Table → upper right corner ⇒ Tree construction

$\underset{1}{A}, \underset{2}{B}, \underset{3}{\underline{\underline{C}}}, \underset{4}{D} \Rightarrow 0.1, 0.2, 0.4, 0.3$

$\underset{i,\ j}{(1, 4)}$ $k = 3$





Knapsack problem and memory Functions

- Knapsack problem – given n items with weight and values. Have to find the subset of items that find the knapsack capacity W with highest value.
- Dynamic Programming – Divide the subsets into 2 categories
  - Don't include i^{th} element → V[i-1, j] , $j-w_i<0$
  - Includes ith element → max{V[i-1, j], v_i + V[i-1,j-w1]}, $j-w_i>=0$

Example – **capacity – W = 5**

Item	1	2	3	4
Weight	2	1	3	2
value	12	10	20	15

- **To find : V[n,W] – maximal value of subset of the n given items that fit the knapsack of capaity 5 →v[4,5]**

Knapsack problem and memory Functions

Item	1	2	3	4
W	2	1	3	2
V	12	10	20	15

V[n,W] → V [4, 5]

Initial Conditions:

1. v[0,j] = 0
2. V[i,0] = 0

Formula:

1. V[i-1, j] , $j-w_i<0$
2. max{V[i-1, j], v_i + V[i-1,j-w1]}, $j-w_i>=0$

Capacity j

i	0	1	2	3	4	5
0	0	0	0	0	0	0
1						
2						
3						
4						

Capacity j

i	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0					
3	0					
4	0					

Item	1	2	3	4
W	2	1	3	2
V	12	10	20	15

W1=2, v1=12

W2=1, v2=10

W3=3, v3=20

W4=2, v4=15

Formula:

1. V[i-1, j] , $j\text{-}w_i<0$

2. max{V[i-1, j], v_i + V[i-1,j-w_i]} , $j\text{-}w_i>=0$

i=1, j=2,wi=w1=2,vi=v1=12

Max{v[0,2], v1+ v[0, 0]}

Max{0, 12+0}

Max{0,12}

12

$j\text{-}w_i$	Formula (i=1)
1-2 = -1	V[0,1] = 0
2-2 = 0	max(V[0,2] , v1+v[0,0])= Max(0,12+0) = max(0,12) =12
3-2 = 1	V[0,3] , v1+v[0,1] = 0,12+0
4-2 = 2	V[0,4] , v1+v[0,2] = 0,12+0
5-2 = 3	V[0,5] , v1+v[0,3] = 0,12+0

Capacity j

i	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0	10	12			
3	0					
4	0					

Item	1	2	3	4
W	2	1	3	2
V	12	10	20	15

W1=2, v1=12

W2=1, v2=10

W3=3, v3=20

W4=2, v4=15

Formula:

1. V[i-1, j] , $j-w_i<0$

2. max{V[i-1, j], v_i + V[i-1,j-w_i]}, $j-w_i>=0$

$j-w_i$	Formula (i=2)
1-1 = 0	Max(V[1,1], 10+v[1,0]) Max(0, 10+0)= max(0,10) =10
2-1 = 1	Max(v[1,2], 10+ v[1,1]) Max(12, 10+0) = 12
3-1 = 2	
4- 1= 3	
5-1 = 4	

Capacity j

i	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0	10	12	22	22	22
3	0	10	12	22	30	32
4	0	10	15	25	30	37

Item	1	2	3	4
W	2	1	3	2
V	12	10	20	15

W1=2, v1=12

W2=1, v2=10

W3=3, v3=20

W4=2, v4=15

Formula:

1. $V[i-1, j]$, $j-w_i<0$

2. $\max\{V[i-1, j], v_i + V[i-1,j-w_i]\}$, $j-w_i>=0$

C[i ,j] = C[4,5]

1. here i=4, j=5

2. j- wi = j-w1 =

Capacity j

i	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	**12**	12	12
2	0	10	12	***22***	22	22
3	0	10	12	22	30	32
4	0	10	15	25	30	**37**

Item	1	2	3	4
W	2	1	3	2
V	12	10	20	15

W1=2, v1=12

W2=1, v2=10

W3=3, v3=20

W4=2, v4=15

Optimal Subset (tracking back the table for finding the subset)

Maximum value V(4,5) = 37

Optimal Subset = {4,2,1}= 15+10+12 = 37

Knapsack Capacity W= 5 → 5-2 =3-1=2-2 = 0