In fuzzy logic, relations between elements of different sets are represented using fuzzy relations. These relations extend classical relations by allowing degrees of membership between elements.

A **fuzzy relation** $R$ between two universes $X$ and $Y$ is defined as:

$$R : X \times Y \to [0, 1]$$

where $R(x, y)$ represents the membership degree of the pair $(x, y)$ in the relation $R$.

## 2. Composition of Fuzzy Relations

Composition of fuzzy relations is used to derive a relationship between elements of different sets through an intermediate set. It is useful in multi-stage decision-making and inference processes.

### 2.1 Definition

Given two fuzzy relations:

- $R$ between sets $X$ and $Y$, denoted as $R : X \times Y$.
- $S$ between sets $Y$ and $Z$, denoted as $S : Y \times Z$.

The composition of $R$ and $S$, denoted as $R \circ S$, results in a fuzzy relation between $X$ and $Z$, i.e., $R \circ S : X \times Z$.

## 2.2 Max-Min Composition

The most commonly used composition method in fuzzy logic is the **Max-Min composition**:

$$(R \circ S)(x, z) = \max_{y \in Y} \min(R(x, y), S(y, z))$$

This means that for each pair $(x, z)$, the relation strength is determined by taking the **minimum** of corresponding membership values from $R$ and $S$, and then choosing the **maximum** among these values.

**Example: Max-Min Composition**

Let:

$$R = \begin{bmatrix} 0.2 & 0.7 \\ 0.5 & 0.9 \end{bmatrix}$$

between sets $X$ and $Y$, and

between sets $X$ and $Y$, and

$$S = \begin{bmatrix} 0.8 & 0.3 \\ 0.6 & 0.5 \end{bmatrix}$$

between sets $Y$ and $Z$.

The composition $R \circ S$ is computed as:

For $(x_1, z_1)$:

$$\max(\min(0.2, 0.8), \min(0.7, 0.6)) = \max(0.2, 0.6) = 0.6$$

For $(x_1, z_2)$:

$$\max(\min(0.2, 0.3), \min(0.7, 0.5)) = \max(0.2, 0.5) = 0.5$$

For $(x_2, z_1)$:

$$\max(\min(0.5, 0.8), \min(0.9, 0.6)) = \max(0.5, 0.6) = 0.6$$

For $(x_2, z_2)$:

$$\max(\min(0.5, 0.3), \min(0.9, 0.5)) = \max(0.3, 0.5) = 0.5$$

Thus, the resulting composition matrix is:

$$R \circ S = \begin{bmatrix} 0.6 & 0.5 \\ 0.6 & 0.5 \end{bmatrix}$$

## 2.3 Max-Product Composition

Another method is the **Max-Product composition**, where:

$$(R \circ S)(x, z) = \max_{y \in Y}(R(x, y) \times S(y, z))$$

Here, the **product** replaces the minimum operation.

#### Example: Max-Product Composition

Using the same matrices $R$ and $S$, the computation is:

For $(x_1, z_1)$:

$$\max(0.2 \times 0.8, 0.7 \times 0.6) = \max(0.16, 0.42) = 0.42$$

For $(x_1, z_2)$:

$$\max(0.2 \times 0.3, 0.7 \times 0.5) = \max(0.06, 0.35) = 0.35$$

For $(x_2, z_1)$:

$$\max(0.5 \times 0.8, 0.9 \times 0.6) = \max(0.4, 0.54) = 0.54$$

For $(x_2, z_2)$:

$$\max(0.5 \times 0.3, 0.9 \times 0.5) = \max(0.15, 0.45) = 0.45$$

Thus, the Max-Product composition matrix is:

$$R \circ S = \begin{bmatrix} 0.42 & 0.35 \\ 0.54 & 0.45 \end{bmatrix}$$

**Applications of Fuzzy Relation Composition**

- **Decision Support Systems:** Helps in modeling multi-stage decision processes.
- **Fuzzy Control Systems:** Used in fuzzy logic controllers for process control.
- **Pattern Recognition:** Useful for classification and clustering problems.
- **Medical Diagnosis:** Aids in reasoning about symptoms and diseases.