In classical set theory, elements either belong to a set ($1$) or they do not ($0$). However, **fuzzy sets** allow partial membership, where each element has a membership value between 0 and 1.

A **fuzzy set** $A$ in a universal set $U$ is defined as:

$$A = \{(x, \mu_A(x)) | x \in U, \mu_A(x) \in [0,1]\}$$

where $\mu_A(x)$ is the **membership function** of element $x$ in $A$.

# 2. Basic Set Operations in Fuzzy Logic

## 2.1 Union (OR Operation)

In fuzzy logic, the union of two fuzzy sets $A$ and $B$ is defined using the **maximum** membership function:

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

**Example**

Let:

$$A = \{(x_1, 0.3), (x_2, 0.6), (x_3, 0.8)\}$$

$$B = \{(x_1, 0.5), (x_2, 0.4), (x_3, 0.7)\}$$

Then:

$$A \cup B = \{(x_1, \max(0.3, 0.5)), (x_2, \max(0.6, 0.4)), (x_3, \max(0.8, 0.7))\}$$

$$= \{(x_1, 0.5), (x_2, 0.6), (x_3, 0.8)\}$$

### 2.2 Intersection (AND Operation)

The intersection of two fuzzy sets is defined using the **minimum** membership function:

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

**Example**

Using the same sets $A$ and $B$:

$$A \cap B = \{(x_1, \min(0.3, 0.5)), (x_2, \min(0.6, 0.4)), (x_3, \min(0.8, 0.7))\}$$

$$= \{(x_1, 0.3), (x_2, 0.4), (x_3, 0.7)\}$$

### 2.3 Complement (NOT Operation)

The complement of a fuzzy set $A$ is defined as:

$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

**Example**

Using the same set $A$:

$$\neg A = \{(x_1, 1 - 0.3), (x_2, 1 - 0.6), (x_3, 1 - 0.8)\}$$

$$= \{(x_1, 0.7), (x_2, 0.4), (x_3, 0.2)\}$$

## 3. Properties of Fuzzy Set Operations

- **Idempotency**: $A \cup A = A$, $A \cap A = A$
- **Commutativity**: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- **Associativity**: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$
- **Distributivity**: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **De Morgan's Laws**:

$$\neg(A \cup B) = \neg A \cap \neg B$$

$$\neg(A \cap B) = \neg A \cup \neg B$$

**Applications of Fuzzy Set Operations**

- **Fuzzy Logic Controllers**: Used in decision-making processes in AI and robotics.
- **Image Processing**: Edge detection, segmentation using fuzzy logic.
- **Medical Diagnosis**: Combining symptoms using fuzzy intersection and union.
- **Decision-Making Systems**: Evaluating multi-criteria problems.