



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) & ;

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & ; B.Tech.IT)



Unit- IV

Complex Integration

Cauchy's integral theorem (or) Cauchy's fundamental theorem :

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve c , then

$$\oint_c f(z) dz = 0$$

Cauchy's integral formula :

If $f(z)$ is analytic inside and on a simple closed curve c and a be any point inside c then

$$\oint_c \frac{f(z) dz}{z-a} = 2\pi i \cdot f(a)$$

Note : If the point lies outside c then $\oint_c \frac{f(z) dz}{z-a} = 0$

Cauchy's integral formula for derivatives :

If $f(z)$ is analytic inside and on a simple closed curve c and a be any point inside c then,

$$\oint_c \frac{f(z)}{(z-a)^2} dz = \frac{2\pi i}{1!} f'(a)$$

$$\oint_c \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(a)$$

In general,

$$\oint_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$



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Problems :-

1) Evaluate $\int_c \frac{\cos \pi z}{z-1} dz$ if c is $|z|=2$.

Sol: Formula:

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$z-a=0$$

$$z-1=0$$

$$z=1$$

Given: $|z|=2$

$$1 < 2$$

1 lies inside c .

$$\int_c \frac{\cos \pi z}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi i \cdot \cos \pi$$

$$= 2\pi i (-1)$$

$$= -2\pi i$$

(Here $a=1$)

$$f(z) = \cos \pi z$$

$$\boxed{\int_c \frac{\cos \pi z}{z-1} dz = -2\pi i}$$

2) Evaluate $\int_c \frac{z dz}{z-2}$ where c is the circle $|z|=1$.

Sol: $f(z) = \frac{z}{z-2}$

$$z-2=0$$

$$z=2$$

($\because |z|=1$)

$$2 > 1$$

Here $z=2$ lies outside c .

$$\int_c f(z) dz = 0$$

$$\boxed{\int_c \frac{z dz}{z-2} = 0}$$

3) Evaluate $\int_c \frac{z}{(z-1)^3} dz$ where c is $|z|=2$.

Sol:

$$z-1=0$$

$$z=1$$

Given: $|z|=2 \Rightarrow 1 < 2$



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$\therefore$ It lies inside C .

Formula:
$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

$$\int_C \frac{f(z)}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1)$$

$$= \frac{2\pi i}{2} f''(1)$$

$$= \pi i f''(1)$$

$$= \pi i f''(0)$$

$$= 0$$

$$\boxed{\int_C \frac{f(z)}{(z-1)^3} dz = 0}$$

$$f(z) = z$$

$$f'(z) = 1$$

$$f''(z) = 0$$

$$f''(1) = 0$$

4) Using Cauchy's integral formula evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$

where C is $|z| = \frac{3}{2}$.

Sol. Using Partial fractions,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow \textcircled{1}$$

$$1 = A(z-2) + B(z-1)$$

Put $z=1 \Rightarrow 1 = A(1-2) + 0$

$$1 = A(-1)$$

$$\boxed{A = -1}$$

Put $z=2 \Rightarrow 1 = A(0) + B(2-1)$

$$1 = 0 + B$$

$$\boxed{B = 1}$$



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$$(1) \Rightarrow \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

∴ Multiply by $\cos \pi z^2$ on both sides, we get

$$\frac{\cos \pi z^2}{(z-1)(z-2)} = \frac{-\cos \pi z^2}{z-1} + \frac{\cos \pi z^2}{z-2}$$

$$\therefore \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{-\cos \pi z^2}{z-1} dz + \int_C \frac{\cos \pi z^2}{z-2} dz$$

→ (2)

$$z=1 \quad ; \quad z=2$$

$$1 < \frac{3}{2} \quad ; \quad 2 > \frac{3}{2}$$

$$\begin{aligned} \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz &= -2\pi i f(1) + 0 \\ &= -2\pi i (-1) \\ &= 2\pi i \end{aligned}$$

$$f(z) = \cos \pi z^2$$

$$\begin{aligned} f(1) &= \cos \pi \\ &= -1 \end{aligned}$$

$$\boxed{\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i}$$

5) Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is $|z|=3$.

Sol: Using Partial fractions,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow (1)$$

$$1 = A(z-2) + B(z-1)$$

$$\text{Put } z=1 \Rightarrow 1 = A(1-2) + B(1-1)$$

$$\boxed{-1 = A}$$



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$$\text{Put } z=2 \Rightarrow 1 = A(2-2) + B(2-1)$$

$$1 = 0 + B$$

$$\boxed{1 = B}$$

$$\textcircled{1} \Rightarrow \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} \left(\frac{-1}{z-1} + \frac{1}{z-2} \right) dz$$

$$= - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz + \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

$$= - \left[2\pi i (0-1) \right] + 2\pi i (0+1)$$

$$\begin{matrix} z=1 \\ z=2 \end{matrix}$$

$$= -(-2\pi i) + 2\pi i$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

$$\boxed{\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 4\pi i}$$