At $z = \frac{\pi}{4}$

$$f(z) = f\left(\frac{\pi}{4}\right) + \frac{(z-\pi/4)}{1!} f'\left(\frac{\pi}{4}\right) + \frac{(z-\pi/4)^2}{2!} f''\left(\frac{\pi}{4}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{\left(z-\frac{\pi}{4}\right)^2}{2}\left(\frac{-1}{\sqrt{2}}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{6}\left(\frac{-1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3 + \cdots$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3 + \cdots$$

Zeros of an analytic function:

If a function $f(z)$, analytic in a region R, is zero at a point $z = z_0$ in R, then z_0 is called a zero of $f(z)$.

Simple zero:

If $f(z_0) = 0$ and $f'(z_0) \neq 0$, then $z = z_0$ is called a simple zero of $f(z)$ or a zero of the first order.

Zero of order n:

If $f(z_0) = f'(z_0) = \cdots = f^{n-1}(z_0) = 0$ and $f^n(z_0) \neq 0$ then z_0 is called zero of order n.

(or)

An analytic function $f(z)$ is said to have a zero of order n if $f(z)$ can be expressed as $f(z) = (z-z_0)^n \phi(z)$, where $\phi(z)$ is analytic and $\phi(z_0) \neq 0$.

Problems:

1) Find the zeros of $f(z) = \frac{z^2+1}{1-z^2}$

Sol: The zeros of $f(z)$ are given by $f(z)=0$

i.e.) $f(z) = \frac{z^2+1}{1-z^2} = \frac{(z+i)(z-i)}{1-z^2} = 0$

$(z+i)(z-i) = 0$

$z = i$ is a simple zero

$z = -i$ is a simple zero

2) Find the zeros of $f(z) = \sin \frac{1}{z-a}$.

Sol: The zeros are given by $f(z)=0$

$\sin \frac{1}{z-a} = 0$

$\frac{1}{z-a} = n\pi, \quad n = \pm 1, \pm 2, \ldots$

$(z-a)n\pi = 1$

The zeros are $z = a + \frac{1}{n\pi}, \quad n = \pm 1, \pm 2, \ldots$

3) Find the zeros of $\frac{z^3-1}{z^3+1}$.

Sol: The zeros of $f(z)$ are given by $f(z) = 0$

$\frac{z^3-1}{z^3+1} = 0$

$z^3 - 1 = 0$

$z = (1)^{1/3} = 1, \omega, \omega^2$ (Cube roots of unity)

4) Find the zeros of $\frac{\sin z - z}{z^3}$

Sol:- let $f(z) = \frac{\sin z - z}{z^3}$

zeros of $f(z)$ are given by $f(z) = 0$

$$f(z) = \frac{\sin z - z}{z^3} = \frac{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots\right] - z}{z^3}$$

$$= \frac{-\frac{z^3}{3!} + \frac{z^5}{5!} - \dots}{z^3}$$

$$= -\frac{1}{3!} + \frac{z^2}{5!} - \dots$$

$$\lim_{z \to 0} \frac{\sin z - z}{z^3} \neq 0$$

$f(z)$ has no zeros.

5) Find the zeros of $\frac{1 - e^{2z}}{z^4}$

Sol:- $f(z) = \frac{1 - e^{2z}}{z^4}$

zeros of $f(z)$ are given by $f(z) = 0$

$$\frac{1 - e^{2z}}{z^4} = 0$$

$$1 - e^{2z} = 0$$

$$e^{2z} = e^{2in\pi}$$

$$2z = 2in\pi$$

$$z = in\pi \quad \Rightarrow 0, \pm\pi, \pm 2\pi, \dots$$

Singularities:.

A Point $z=z_0$ at which a function $f(z)$ fails to be analytic is called a Singular Point or singularity of $f(z)$.

Eg:. $f(z)=\frac{1}{z-3}$

Here $z=3$ is a singular Point of $f(z)$.

Types of Singularities:.

i) Isolated Singularity:.

A Point $z=z_0$ is said to be isolated singularity of $f(z)$ if

i) $f(z)$ is not analytic at $z=z_0$

ii) There exists a neighbourhood of $z=z_0$ containing no other singularity.

Eg:. $f(z)=\frac{1}{z}$

It is analytic everywhere except at $z=0$.

$z=0$ is an isolated singularity.

Note:. If $z=z_0$ is an isolated singular Point of a function $f(z)$, then the singularity is called

(i) a removable singularity (or) (ii) a pole (or)

(iii) an essential singularity

According to Laurent's Series about $z=z_0$ of $f(z)$ valid in a deleted neighbourhood of $z=z_0$ has

i) no negative powers or

ii) a finite number of negative powers or

iii) an infinite number of negative powers.

Note:2 Let $z = z_0$ be an isolated singular point of the function $f(z)$. Then $f(z)$ is analytic inside and on a circle c except at $z = z_0$, the centre of c and therefore $f(z)$ can be expanded by a Laurent's series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} b_n (z-z_0)^{-n}$$

Removable Singularity:

A singular point $z = z_0$ is called a removable singularity of $f(z)$ if $\lim_{z \to z_0} f(z)$ exists finitely.

$$f(z) = \frac{\sin z}{z}$$

$$\frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} - \cdots\right]$$

$$= 1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} - \cdots$$

there is no negative powers of z.

$z = 0$ is a removable singularity of $f(z)$.

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\sin z}{z} = 1 \quad \text{(finite)}$$

$z = 0$ is a removable singularity.

Problems:

1) what is the nature of the singularity $z=0$ of the function $f(z)=\frac{\sin z - z}{z^3}$

Sol: $f(z)=\frac{\sin z - z}{z^3}$

A function $f(z)$ is not defined at $z=0$.

By L'Hospital's rule, $\lim_{z\to 0}\frac{\sin z - z}{z^3} = \lim_{z\to 0}\frac{\cos z - 1}{3z^2} = \lim_{z\to 0}\frac{-\sin z}{6z}$

$$= \lim_{z\to 0}\frac{-\cos z}{6}$$

$$= -\frac{1}{6}$$

Since the limit exists and finite, the singularity at $z=0$ is a removable singularity.

2) Find the nature of the singularity at $z=0$ of $f(z)=\frac{\sin z}{z}$.

Sol: $f(z)=\frac{\sin z}{z}$

A function $f(z)$ is not defined at $z=0$

By L'Hospital's rule,

$$\lim_{z\to 0}\frac{\sin z}{z} = \lim_{z\to 0}\frac{\cos z}{1} = 1$$

Since the limit exists and is finite, the singularity at $z=0$ is a removable singularity.