$$= \frac{1}{z} + \frac{2}{z\left(1-\frac{2}{z}\right)} - \frac{3}{z\left(1+\frac{1}{z}\right)}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right) - \frac{3}{z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2 \ldots\right)$$

iv) $1<|z+1|<3$

Let $u = z+1$

$z = u-1$

$1<|u|<3$

$\frac{1}{|u|}<1, \quad \left|\frac{u}{3}\right|<1$

$$f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{2}{-3\left(1-\frac{u}{3}\right)} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\cdots\right) - \frac{2}{3}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\cdots\right) - \frac{3}{u}$$

$$f(z) = \frac{1}{z+1}\left(1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\cdots\cdots\right) - \frac{2}{3}\left(1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\cdots\right) - \frac{3}{z+1}$$

Residues:

If $z = z_0$ is an isolated Singular point of $f(z)$, we can find the Laurent Series of $f(z)$ about $z = z_0$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

The coefficient of $\frac{1}{z-z_0}$ in the above expansion is called the residue of $f(z)$ at $z = z_0$.

ie,) b_1 is the residue of $f(z)$ at $z = z_0$.

From a definition of b_n,

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

$$b_1 = \frac{1}{2\pi i} \int_C f(z) dz$$

Residue of $f(z)$ at $z = z_0$ may be denoted by Res $[f(z), z_0]$.

Evaluation of Residues:

i) Residue at a Pole of order m:

If $z = z_0$ is a Pole of order m, a simple formula to determine the residue is given by

$$b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

ii) Residue at a Simple Pole:

If $z = z_0$ is a Simple Pole

If $m=1$, the residues to

$$b_1 = \lim_{z \to z_0} (z-z_0) f(z)$$

iv If $z=z_0$ is a simple Pole of $f(z)$, then

$$\text{Res}\left[f(z), z_0\right] = \lim_{z\to z_0} (z-z_0) f(z).$$

iii) If $z=z_0$ is a simple pole of $f(z)$ and if

$$f(z) = \frac{\phi(z)}{\phi(z)} \text{ then } \text{Res}\left[f(z), z_0\right] = \frac{\phi(z_0)}{\psi'(z_0)}$$

Problems:

1) Find the residue of $f(z) = \frac{z}{(z-1)^2}$ at its Pole.

Sol: Given $f(z) = \frac{z}{(z-1)^2}$

Here $z=1$ is a Pole of order 2.

$$Re(z=z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

$$Re(z=1) = \lim_{z\to 1} \frac{1}{1!} \frac{d}{dz}\left(\cancel{(z-1)^2} \frac{z}{\cancel{(z-1)^2}}\right)$$

$$= \lim_{z\to 1} \frac{d}{dz}(z)$$

$$= \lim_{z\to 1}(1)$$

$$= 1$$

2) Find the residue of cot z at z = 0.

Sol: $f(z) = \cot z = \frac{\cos z}{\sin z} = \frac{\phi(z)}{\psi(z)}$

Here z = 0 is a simple pole.

$\phi(z) = \cos z$ $\qquad \psi(z) = \sin z$

$\phi(0) = \cos 0 = 1$ $\qquad \psi'(z) = \cos z$

$\psi'(0) = 1$

$$Re[f(z), 0] = \frac{\phi(0)}{\psi'(0)} = \frac{1}{1} = 1.$$

3) Find the residue of $f(z) = \frac{1}{z^2 e^z}$

Sol: $f(z) = \frac{e^{-z}}{z^2} = \frac{1 - z + \frac{z^2}{\lfloor 2} - \frac{z^3}{\lfloor 3} + \cdots}{z^2}$

$$= \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \cdots$$

$Re(f(z), 0) =$ Coefficient of $\frac{1}{z}$ in Laurent expansion

$Re(f(z), 0) = -1$

4) Find the residues at z = 0 of the functions

i) $f(z) = e^{1/z}$ $\qquad$ ii) $f(z) = \frac{\sin z}{z^4}$ $\qquad$ iii) $f(z) = z \cos \frac{1}{z}$.

Sol: The residues are the coefficients of $\frac{1}{z}$ in the Laurent's expansion of f(z) about z = 0.

(i.e.) i) $e^{\frac{1}{z}} = 1 + \frac{\frac{1}{z}}{\lfloor 1} + \frac{\left(\frac{1}{z}\right)^2}{\lfloor 2} + \cdots$

$$= 1 + \frac{1}{\lfloor 1}\left(\frac{1}{z}\right) + \frac{1}{\lfloor 2}\frac{1}{z^2} + \frac{1}{\lfloor 3}\frac{1}{z^3} + \cdots$$