



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)



Cauchy-Residue theorem:

If $f(z)$ is analytic at all points inside and on a simple closed curve C except at a finite number of points $z_1, z_2, \dots, z_n$ inside C , then $\int_C f(z) dz = 2\pi i$ (Sum of residues of $f(z)$ at $z_1, z_2, \dots, z_n$)

Proof:

Given that $f(z)$ is not analytic only at $z_1, z_2, \dots, z_n$.

Draw the non-intersecting small circles $C_1, C_2, \dots, C_n$ with centre at $z_1, z_2, \dots, z_n$ and radii $\rho_1, \rho_2, \dots, \rho_n$ lying wholly inside C .

Then $f(z)$ is analytic in the region between C and $C_1, C_2, \dots, C_n$.

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz + \dots + \int_{C_n} f(z) dz \quad \rightarrow (1)$$

Now, $z_1, z_2, \dots, z_n$ are the singular points of $f(z)$.

$\{ \text{Res } f(z) \}_{z=z_i}$ = the coef of $\frac{1}{z-z_i}$ in the Laurent's Series of $f(z)$ about $z=z_i$ (by def of residues)

$$= b_{-1} = \frac{1}{2\pi i} \int_{C_i} \frac{f(z)}{(z-z_i)^{-1-i}} dz$$

$$\text{Since } b_n = \frac{1}{2\pi i} \int_{C_i} \frac{f(z)}{(z-z_i)^{1-n}} dz$$

$$= \frac{1}{2\pi i} \int_{C_i} \frac{f(z)}{z-z_i} dz$$

$$= \frac{1}{2\pi i} \int_{C_i} f(z) dz$$



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)



$$\int_C f(z) dz = 2\pi i \left\{ \text{Res } f(z) \right\}_{z=z_1} \rightarrow (2)$$

From (1) & (2),

$$\begin{aligned} \int_C f(z) dz &= 2\pi i \left\{ \text{Res } f(z) \right\}_{z=z_1} + 2\pi i \left\{ \text{Res } f(z) \right\}_{z=z_2} \\ &\quad + \dots + 2\pi i \left\{ \text{Res } f(z) \right\}_{z=z_n} \\ &= 2\pi i \left\{ [\text{Res } f(z)]_{z=z_1} + [\text{Res } f(z)]_{z=z_2} + \dots \right. \\ &\quad \left. + [\text{Res } f(z)]_{z=z_n} \right\} \\ &= 2\pi i \left\{ \text{Sum of residues of } f(z) \text{ at } z=z_1, z_2, \dots, z_n \right\} \\ &= 2\pi i \sum_i R_i \end{aligned}$$

Problems:

1) Evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2} dz$ where C is the circle $|z|=3$.

Sol: Let $f(z) = \frac{e^z}{(z+2)(z+1)^2}$

The Poles of $f(z)$ are given by

$$(z+2)(z+1)^2 = 0$$

$z = -1$ is a pole of order 2.

$z = -2$ is a pole of order 1.

Given: $|z|=3$

$z = -1 \Rightarrow |z|=1 < 3$, lies inside C .



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)



$z = -2 \Rightarrow |z| = 2 < 3$, lies inside by c .

$$\{ \text{Res } f(z) \}_{z=-2} = \lim_{z \rightarrow -2} \frac{(z+2) e^z}{(z+1)^2}$$

$$= \frac{e^2}{(-2+1)^2}$$

$$= e^2$$

$$\{ \text{Res } f(z) \}_{z=-1} = \lim_{z \rightarrow -1} \frac{1}{1!} \frac{d}{dz} \left(\frac{(z+1)^2 e^z}{(z+1)^2 (z+2)} \right)$$

$$= \lim_{z \rightarrow -1} \frac{d}{dz} \frac{e^z}{(z+2)}$$

$$= \lim_{z \rightarrow -1} \frac{-e^z}{(z+2)^2} = \frac{-e^2}{(-1+2)^2} = -e^2$$

By Cauchy's residue theorem,

$$\oint_c f(z) dz = 2\pi i \left(\text{sum of the residues of } f(z) \text{ at the poles which lie inside } c \right)$$

$$\oint_c \frac{e^z dz}{(z+2)(z+1)^2} = 2\pi i (e^2 - e^2)$$

$$= 0$$

$$\boxed{\oint_c \frac{e^z dz}{(z+2)(z+1)^2} = 0}$$



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)



2) Evaluate $\int_c \frac{4-3z}{z(z-1)(z-2)} dz$ where c is the circle

$$|z| = \frac{3}{2}$$

sol: Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$

The Poles of $f(z)$ are

$$z(z-1)(z-2) = 0$$

$z=0, z=1, z=2$ are Poles of order 1.

$z=0 \Rightarrow |z|=0 < \frac{3}{2}$ lies inside c .

$z=1 \Rightarrow |z|=1 < \frac{3}{2}$ lies inside c .

$z=2 \Rightarrow |z|=2 > \frac{3}{2}$ lies outside c .

$$\begin{aligned} \{ \text{Res } f(z) \}_{z=0} &= \lim_{z \rightarrow 0} (z-0) \frac{4-3z}{z(z-1)(z-2)} \\ &= \frac{4}{(-1)(-2)} = 2 \end{aligned}$$

$$\begin{aligned} \{ \text{Res } f(z) \}_{z=1} &= \lim_{z \rightarrow 1} (z-1) \frac{4-3z}{z(z-1)(z-2)} \\ &= \frac{1}{-1} = -1 \end{aligned}$$

$$\{ \text{Res } f(z) \}_{z=2} = 0 \text{ (lies outside } c \text{)}$$

By Cauchy's residue theorem,

$$\begin{aligned} \int_c f(z) dz &= 2\pi i \text{ (Sum of residues of } f(z) \text{ at the poles which lie inside } c) \\ &= 2\pi i (2-1) = 2\pi i \end{aligned}$$