Type – II Clairut's type

$$z = px + qy + f(p, q)$$

1. Solve $z = px + qy + pq$

Soln → Given $z = px + qy + pq \rightarrow (1)$

The given PDE is clairut's type

(i) To find the Complete integral

put $p = a$, $q = b$ in (1), we get

$$z = ax + by + ab \rightarrow (2)$$

which is the required complete integral.

(ii) To find the singular integral

Diff (2) p.d.w.r.to a, we get

$$x + 0 + b = 0$$
$$b = -x$$

Diff (2) p.d.w.r.to b, we get

$$0 + y + a = 0$$
$$a = -y$$

Subst the values of a, b in (2), we get

$$z = -xy - \cancel{xy} + \cancel{xy}$$
$$z = -xy$$

which is the required singular integral.

2. Solve $z = px+qy+p^2+pq+q^2$.

$\rightarrow$ Given $z = px+qy+p^2+pq+q^2 \longrightarrow (1)$

The given PDE is clairuts type.

(i) To find the complete integral

put $p=a$, $q=b$ in (1), we get

$z = ax+by+a^2+ab+b^2 \longrightarrow (2)$

which is the required Complete integral.

(ii) To find the singular integral

Diff (2), p.w.r. to a, we get

$x+2a+b=0$

$2a+b = -x \longrightarrow (3)$

Diff (2), p.w.r. to b, we get

$y+a+2b=0$

$a+2b = -y \longrightarrow (4)$

Solve (3) & (4),

(4) $\times 2 \Rightarrow 2a+4b = -2y \longrightarrow (5)$

(3) $\Rightarrow 2a+b = -x \longrightarrow (3)$

(5) $-$ (3) $\Rightarrow 3b = -2y+x$

$b = \dfrac{x-2y}{3}$

(3) $\times 2 \Rightarrow 4a+2b = -2x \longrightarrow (6)$

(4) $\Rightarrow a+2b = -y \longrightarrow (4)$

(6) $-$ (4) $\Rightarrow 3a = -2x+y$

$a = \dfrac{y-2x}{3}$

Sub the values of a, b in (2), we get

$$z = \left(\frac{y-2x}{3}\right)x + \left(\frac{x-2y}{3}\right)y + \left(\frac{y-2x}{3}\right)^2 + \left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right) + \left(\frac{x-2y}{3}\right)^2$$

$$= \frac{xy-2x^2}{3} + \frac{xy-2y^2}{3} + \frac{y^2+4x^2-4xy}{9} + \frac{xy-2y^2-2x^2+4xy}{9} + \frac{x^2+4y^2-4xy}{9}.$$

$$= \frac{3xy-6x^2+3xy-6y^2+y^2+4x^2-4xy+xy-2x^2-2y^2+4xy+x^2+4y^2-4xy}{9}$$

$$= \frac{3xy-3x^2-3y^2}{9}$$

$$z. = \frac{xy-x^2-y^2}{3}$$

(iii) To find the general integral

put $b = f(a)$ in ②, we get

$$z = ax + f(a)y + a^2 + af(a) + [f(a)]^2 \quad \text{—— ⑦}$$

$$\frac{\partial z}{\partial a} = 0.$$

3. Find singular solution of $z = px + qy + \sqrt{1+p^2+q^2}$

<u>Sol</u>

Gn: $z = px + qy + \sqrt{1+p^2+q^2}$ —— ①

The given PDE is Clairut's type.

(i) To find the complete integral

put $p=a$, $q=b$ in ①, we get

$$z = ax + by + \sqrt{1+a^2+b^2} \quad \text{—— ②}$$

which is the required complete integral

(ii) To find the singular integral

Diff ② partially w.r to a, we get

$$x + \frac{1}{2\sqrt{1+a^2+b^2}}(2a) = 0$$

$$x = \frac{-a}{\sqrt{1+a^2+b^2}} \quad \text{③}$$

Diff ② partially w.r to b, we get

$$y + \frac{1}{2\sqrt{1+a^2+b^2}}(2b) = 0$$

$$y = \frac{-b}{\sqrt{1+a^2+b^2}} \quad \text{④}$$

$$\text{③} = x^2 = \frac{a^2}{1+a^2+b^2} \quad \text{⑤}$$

$$\text{④} = y^2 = \frac{b^2}{1+a^2+b^2} \quad \text{⑥}$$

$$\text{⑤} + \text{⑥} \Rightarrow x^2+y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$= \frac{1+a^2+b^2-1}{1+a^2+b^2} \quad \text{[Add \& sub 1 in numerator]}$$

$$= \frac{1+a^2+b^2}{1+a^2+b^2} - \frac{1}{1+a^2+b^2}$$

$$x^2+y^2 = 1 - \frac{1}{a^2+b^2+1}$$

$$\frac{1}{a^2+b^2+1} = 1 - x^2 - y^2$$

$$\frac{1}{1-x^2-y^2} = 1+a^2+b^2$$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}} \quad \text{⑦}$$

Sub ⑦ in ③, we get

$$③ \Rightarrow a = \frac{-x}{\sqrt{1-x^2-y^2}}$$

Sub ⑦ in ④, we get.

$$④ \Rightarrow b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Sub the a, b in ②, we get.

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} + \frac{-y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$$x^2+y^2+z^2 = 1$$

4. Solve $z = px + qy + p^2q^2$. (H.W).