Lagrange's linear PDE.

1. Solve $xp + yq = z$.

Soln

Given $xp + yq = z$.

The given PDE is a lagrange's linear equation with

$P = x, Q = y, R = z$.

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

Taking the 1st two ratios, we get.

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get:

$$\log x = \log y + \log c,$$

$$\log x - \log y = \log c,$$

$$\log\left(\frac{x}{y}\right) = \log c,$$

$$\frac{x}{y} = c,$$

$$u = \frac{x}{y}$$

Taking the last two ratios, we get.

$$\frac{dy}{y} = \frac{dz}{z}.$$

Integrating we get,

$$\log y = \log z + \log c_2$$
$$\log y - \log z = \log c_2$$
$$\log (y/z) = \log c_2$$
$$\frac{y}{z} = c_2$$
$$v = y/z.$$

The general solution of the given equation $f(u,v) = 0$

$$f\left(\frac{x}{y}, \frac{y}{z}\right) = 0.$$

2. Solve $xp + yq = x$.

Sol: The given PDE is a Lagrange's linear equation with $P = x$, $Q = y$, $R = x$

The subsidiary equation are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x}$$

Taking the 1st two ratio, we get.

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get.

$$\log x = \log y + \log c_1$$
$$\log x - \log y = \log c_1$$
$$\log\left(\frac{x}{y}\right) = \log c_1$$
$$\frac{x}{y} = c_1$$
$$u = x/y$$

Taking the fiist and third ratios, we get

$$\frac{dx}{x} = \frac{dz}{x}$$

$$dx = dz.$$

Integrating we get.

$$x = z + c_2$$

$$c_2 = x - z$$

$$v = x - z.$$

The general solution of the given equation is $f(u,v) = 0$

$$f\left(\frac{x}{y}, x - z\right) = 0$$

3. Solve $p\tan x + q\tan y = \tan z$.

Sol

Given $p\tan x + q\tan y = \tan z$.

The given PDE is a Lagrange's linear equation with

$P = \tan x \quad Q = \tan y \quad R = \tan z.$

The subsidiary equation are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}.$$

Taking the 1st two ratios, we get

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\cot x\, dx = \cot y\, dy.$$

Integrating, we get

$$\log(\sin x) = \log(\sin y) + \log c_1$$

$$= \log(\sin y c_1)$$

$$\sin x = \sin y c_1$$

$$\frac{\sin x}{\sin y} = c_1$$

$u = \frac{\sin x}{\sin y}$

Taking the last two ratio, we get.

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}.$$

$$\cot y\,dy = \cot z\,dz.$$

Integrating, we get.

$$\log(\sin y) = \log(\sin z) + \log C_2$$

$$= \log(\sin z\, C_2)$$

$$\sin y = \sin z\, C_2$$

$$\frac{\sin y}{\sin z} = C_2.$$

$$v = \frac{\sin y}{\sin z}.$$

The general solution of the given eqn is $f(u,v)$

i.e., $f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0.$