



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)

COIMBATORE-641 035, TAMIL NADU



UNIT 3 PARTIAL DIFFERENTIAL EQUATIONS

Linear partial differential equations of second order with constant coefficients of homogeneous types

Linear PDE of 2nd order with constant coefficients

Homogeneous Linear PDE's:

A Linear PDE with constant coefficients in which all the partial derivatives are of the same order is called homogeneous, otherwise it is called non-homogeneous

Example:

Homogeneous Equation:-

$$\frac{\partial^2 z}{\partial x^2} + 5 \frac{\partial^2 z}{\partial x \partial y} + 6 \frac{\partial^2 z}{\partial y^2} = \sin x.$$

Non Homogeneous Equation:

$$\frac{\partial^2 z}{\partial x^2} - 5 \frac{\partial z}{\partial x} + 7 \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} = e^{x+y}$$

Notation: $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

Method of finding Complementary function (C.F):

Let the given equation be of the form

$$f(D, D') z = f(x, y)$$

Put $D = m$, $D' = 1$

$$f(m, 1) = 0 \Rightarrow a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$$



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General solution is $y = CF + PI$

RHS=0 ($z = CF$)

1. Solve $(D^2 - 6DD' + 9D'^2)z = 0$

Put $D=m$, $D'=1$

The auxiliary equation is,

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0$$

$$m = 3, 3 \text{ (equal)}$$

The solution is $z = CF$
 $= f_1(y+3x) + x f_2(y+3x)$

Type-I

1. Solve $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

The auxiliary equation is,

$$m^2 - 5m + 6 = 0$$

$$(m-3)(m-2) = 0$$

$$m = 2, 3$$

$$CF = f_1(y+2x) + f_2(y+3x)$$

$$PI = \frac{1}{D^2 - 5DD' + 6D'^2} e^{x+y}$$

$$= \frac{1}{1-5+6} e^{x+y}$$

$$= \frac{1}{2} e^{x+y}$$

Replace
 $D \rightarrow a = 1$
 $D' \rightarrow b = 1$

The solution is $z = CF + PI$
 $= f_1(y+2x) + f_2(y+3x) + \frac{e^{x+y}}{2}$



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3. Type-1 Solve: $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0.$$

$$m = 2, 2 \text{ (equal)}$$

$$CF = f_1(y+2x) + x f_2(y+2x)$$

$$PI = \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$$

$$= \frac{1}{2^2 - 4(2)(1) + 4(1)^2} e^{2x+y}$$

$$= \frac{1}{4 - 8 + 4} e^{2x+y}$$

$$= x \frac{1}{2D - 4D'} e^{2x+y} = x \frac{1}{2(2) - 4(1)} e^{2x+y}$$

$$= x^2 \frac{e^{2x+y}}{2} = \frac{x^2}{2} e^{2x+y}$$

Replace

$$D \rightarrow a = 2$$

$$D' \rightarrow b = 1$$

The solution is $z = CF + PI$

$$= f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$$

4. Solve: $2 \frac{\partial^2 z}{\partial x^2} + 5 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = 0$

$$\text{Given: } (2D^2 + 5DD' + 2D'^2)z = 0.$$

$$\text{A.E is } 2m^2 + 5m + 2 = 0$$

$$2m^2 + 4m + m + 2 = 0$$

$$2m(m+2) + 1(m+2) = 0$$

$$(2m+1)(m+2) = 0 \quad m_1 = -\frac{1}{2}, m_2 = -2$$

→ roots are different



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$$CF \Rightarrow Z = f_1(y - \frac{1}{2}x) + f_2(y - 2x)$$

$$PI \Rightarrow PI = 0.$$

$$\therefore \text{solution is } Z = CF + PI$$

$$= f_1(y - \frac{1}{2}x) + f_2(y - 2x)$$

To find Particular Integral (PI)

$$\text{Type I: } Rhs = f(x, y) = e^{ax+by}$$

$$PI = \frac{1}{\phi(D, D')} e^{ax+by}$$

Replace $D \rightarrow a$, $D' \rightarrow b$

$$\text{then } PI = \frac{1}{\phi(a, b)} e^{ax+by}, \text{ provided } \phi(a, b) \neq 0$$

If $\phi(a, b) = 0$ then differentiate the Denominator w.r.t 'D' and multiply by x in Numerator.

$$2. \text{ Solve! } (2D^2 - 2DD' + D'^2)Z = 2e^{3y} + e^{x+y}.$$

$$AE \text{ is } 2m^2 - 2m + 1 = 0$$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(1)}}{2(2)} = \frac{2 \pm \sqrt{4-8}}{4} = \frac{2 \pm 2i}{4} = \frac{1 \pm i}{2}$$

$$m = \frac{1}{2} \pm \frac{1}{2}i$$

$\therefore$ The roots are imaginary

$$CF \text{ is } Z = f_1(y + x(\frac{1}{2} + \frac{1}{2}i)) + f_2(y + (\frac{1}{2} - \frac{1}{2}i)x)$$

$$PI \Rightarrow PI_1 = \frac{1}{2D^2 - 2DD' + D'^2} 2e^{3y} \quad \rightarrow D=0 \text{ and } D'=3$$

$$= \frac{1}{2(0) - 2(0)(3) + (3)^2} 2e^{3y} = \frac{2}{9} e^{3y}$$



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$$PI_2 = \frac{1}{2D^2 - 2DD' + D'^2} e^{x+y} \quad \Rightarrow D=1, D'=1$$

$$= \frac{1}{2(1)^2 - 2(1)(1) + (1)^2} e^{x+y}$$

$$= e^{x+y}$$

$\therefore$ The solution is $Z = CF + PI$

$$Z = f_1(y + (\frac{1}{2} + \frac{1}{2}i)) + f_2(y + (\frac{1}{2} - \frac{1}{2}i)) + \frac{2}{9} e^{3y} + e^{x+y}$$

4. Solve: $(D^2 - 3DD' + 2D'^2)Z = e^{3x+2y}$

AE is $m^2 - 3m + 2 = 0$

$$(m-2)(m-1) = 0$$

$$m_1 = 1, m_2 = 2$$

CF is $Z = f_1(y+x) + f_2(y+2x)$

$$PI = \frac{1}{D^2 - 3DD' + 2D'^2} e^{3x+2y} \quad D=3, D'=2$$

$$= \frac{1}{(3)^2 - 3(3)(2) + 2(2)^2} e^{3x+2y}$$

$$= \frac{1}{9 - 18 + 8} e^{3x+2y} = -e^{3x+2y}$$

$\therefore$ The solution is $Z = CF + PI$

$$= f_1(y+x) + f_2(y+2x) - e^{3x+2y}$$

5. Solve: $(D^2 - DD' - 2D'^2)Z = e^{5x+y}$

AE is $m^2 - m - 2 = 0$

$$(m-5)(m+4) = 0 \quad m_1 = -4, m_2 = 5$$

$\therefore$ CF is $Z = f_1(y-4x) + f_2(y+5x)$



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$$\begin{aligned}
 \text{5. } PI &= \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y} & D=5, D'=1 \\
 &= \frac{1}{(5)^2 - (5)(1) - 20(1)^2} e^{5x+y} = \frac{1}{25 - 5 - 20} e^{5x+y} \\
 &= \frac{x}{2D - D'} e^{5x+y} = \frac{x}{2(5) - 1} e^{5x+y} \\
 &= \frac{x}{9} e^{5x+y}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{The solution is } z &= CF + PI \\
 &= f_1(y-4x) + f_2(y+5x) + \frac{x e^{5x+y}}{9}
 \end{aligned}$$

$$\begin{aligned}
 \text{6. Solve: } (D^2 + 2DD' + D'^2)z &= e^{x-y} \\
 \text{AE is } m^2 + 2m + 1 &= 0 \Rightarrow (m+1)(m+1) = 0. \\
 m &= -1, -1 \quad \text{Roots are equal}
 \end{aligned}$$

$$\therefore CF \text{ is } z = f_1(y-x) + x f_2(y-x)$$

$$\begin{aligned}
 PI &= \frac{1}{D^2 + 2DD' + D'^2} e^{x-y} & D=1, D'=-1 \\
 &= \frac{1}{(1)^2 + 2(1)(-1) + (-1)^2} e^{x-y} = \frac{e^{x-y}}{1 - 2 + 1} \\
 &= \frac{x}{2D + 2D'} e^{x-y} = \frac{x}{2(1) + 2(-1)} e^{x-y} \\
 &= \frac{x^2}{2} e^{x-y}
 \end{aligned}$$

$$\begin{aligned}
 \text{The solution is } z &= CF + PI \\
 &= f_1(y-x) + x f_2(y-x) + \frac{x^2}{2} e^{x-y}
 \end{aligned}$$