Fourier series

Some Basic formulas :

1. $\int \sin x \, dx = -\cos x$
2. $\int \cos x \, dx = \sin x$
3. $\sin 0 = 0$
4. $\sin \pi/2 = 1$
5. $\sin n\pi = 0$; $\sin (n+1)2\pi = 0$; $\sin (n+1)\pi = 0$
6. $\cos 0 = 1$
7. $\cos \pi/2 = 0$
8. $\cos n\pi = (-1)^n$.
9. $\cos (n+1)2\pi = 1$
10. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
11. $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
12. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
13. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
14. Bernoulli's formula :
$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$
15. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$
16. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

Periodic function :

A function $f(x)$ is said to be periodic with period 'p', if for all x, $f(x+P) = f(x)$ where p is a positive constant, the least value of $P>0$ which is called the period of $f(x)$.

Eg: $f(x) = \sin x = \sin(x+2\pi) = \sin(x+4\pi) = \dots$

So, $\sin x$ is a period of 2π

Dirichlet's condition :

Any function $f(x)$ can be developed as fourier series $\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \cos nx]\right]$ where a_0, a_n and b_n are constants, provided.

i). $f(x)$ is periodic, single valued and finite.

ii). $f(x)$ has a finite no. of finite discontinuities and no infinite discontinuity.

iii). $f(x)$ has atmost a finite number of maxima and minima.

Fourier Series :

A function $f(x)$ is periodic and satisfies Dirichlet's conditions, then it can be represented by an infinite series is called the fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where a_0, a_n and b_n are fourier coefficients.

Euler's Formula:

If a function $f(x)$ defined in $c < x < c+2\pi$ can be expanded as the infinite trigonometric series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$; $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx\,dx$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx\,dx$$

Problems based on Bernoulli's formula:

1]. Find $\int_0^{2\pi} x \sin x\,dx$.

Soln.:

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$u = x$	$v = \sin x$
$u' = 1$	$v_1 = -\cos x$
$u'' = 0$	$v_2 = -\sin x$

Now, $\int_0^{2\pi} x \sin x\,dx$

$$= \left[x(-\cos x) - 1(-\sin x) + 0\right]_0^{2\pi}$$

$$= \left[-x\cos x + \sin x\right]_0^{2\pi}$$

$$= [(-2\pi \cos 2\pi + \sin 2\pi) - 0] \qquad \because \cos 2\pi = 1,\ \sin 2\pi = 0$$

$$\int_0^{2\pi} x \sin x\,dx = -2\pi$$

2]. Evaluate $\int_{-\pi}^{2\pi} (x + x^2) \cos nx\,dx$

Soln.:

$u = x + x^2$	$v = \cos nx$
$u' = 1 + 2x$	$v_1 = \frac{\sin nx}{n}$
$u'' = 2$	$v_2 = -\frac{\cos nx}{n^2}$
$u''' = 0$	$v_3 = -\frac{\sin nx}{n^3}$

Now

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$$\int_{-\pi}^{\pi} (x+x^2)\cos nx\,dx$$

$$= \left[(x+x^2)\frac{\sin nx}{n} - (1+2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right) - 0\right]_{-\pi}^{\pi}$$

$$= \left[(x+x^2)\frac{\sin nx}{n} + (1+2x)\frac{\cos nx}{n^2} - 2\,\frac{\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \left[\left(0 + (1+2\pi)\frac{\cos n\pi}{n^2} - 0\right) - \left(0 + (1-2\pi)\frac{\cos(-n\pi)}{n^2} - 0\right)\right]$$

$$= (1+2\pi)\frac{(-1)^n}{n^2} - (1-2\pi)\frac{(-1)^n}{n^2}$$

$$= (1+2\pi-1+2\pi)\frac{(-1)^n}{n^2}$$

$$= 4\pi\frac{(-1)^n}{n^2}$$

$\cos n\pi = (-1)^n$

$\cos(-n\pi) = \cos n\pi = (-1)^n$

Hw. 1. $\int_0^{2\pi} x^2 \cos nx\,dx$

Problems on $(0, 2\pi)$

Formula:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right]$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

1. Determine the Fourier series for $f(x) = x^2$, $(0, 2\pi)$

Soln.:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right] \quad \rightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}\left[8\pi^3 - 0\right]$$

$$a_0 = \frac{8}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$u = x^2$, $u' = 2x$, $u'' = 2$, $u''' = 0$

$v = \cos nx$, $v_1 = \sin nx/n$, $v_2 = -\cos nx/n^2$, $v_3 = -\sin nx/n^3$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right) - 0\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2\sin nx}{n} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(0 + \frac{4\pi(1)}{n^2} - 0\right) - (0)\right]$$

$\because \sin 2n\pi = 0$, $\cos 2n\pi = 1$, $\sin 0 = 0$

$$a_n = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$u = x^2$, $u' = 2x$, $u'' = 2$, $u''' = 0$

$v = \sin nx$, $v_1 = -\cos nx/n$, $v_2 = -\sin nx/n^2$, $v_3 = \cos nx/n^3$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right) - 0\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\frac{\cos nx}{n} + 2x\frac{\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{4\pi^2}{n} + 0 + \frac{2}{n^3}\right) - \left(0 + 0 + \frac{2}{n^3}\right)\right]$$

$\because \cos 2n\pi = 1$, $\sin 2n\pi = 0$, $\cos 0 = 1$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$b_n = -\frac{4\pi}{n}$$

$$\therefore (1) \Rightarrow \quad f(x) = \frac{8/3\,\pi^2}{2} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right]$$

$$= \frac{4}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

2. If $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $(0, 2\pi)$, determine fourier series for the function $f(x)$.

Soln. :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right] \longrightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = -\frac{1}{12\pi}\left[-\pi^3 - \pi^3\right]$$

$$= -\frac{1}{12\pi}(-2\pi^3)$$

$$a_0 = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nx \, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx \, dx \qquad \int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$$

$$\begin{array}{l|l} u = (\pi - x)^2 & v = \cos nx \\ u' = -2(\pi - x) & v_1 = \sin nx / n \\ u'' = -2(-1) = 2 & v_2 = -\cos nx / n^2 \\ u''' = 0 & v_3 = -\sin nx / n^3 \end{array}$$

$$= \frac{1}{4\pi}\Big[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - \left(-2(\pi - x)\right)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right) - 0\Big]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - 2(\pi - x)\left(\frac{\cos nx}{n^2}\right) - 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$\because \sin 2n\pi = 0$, $\cos 2n\pi = 1$, $\cos 0 = 1$

$$= \frac{1}{4\pi}\left[\left(0 - \frac{2(-\pi)}{n^2} - 0\right) - \left(0 - \frac{2\pi}{n^2} - 0\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right]$$

$$a_n = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \sin nx \, dx \qquad \int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx \, dx$$

$$\begin{array}{l|l} u = \left(\frac{\pi - x}{2}\right)^2 & v = \sin nx \\ & v_1 = -\cos nx / n \\ u' = -2(\pi - x) & v_2 = -\sin nx / n^2 \\ u'' = -2(-1) = 2 & v_3 = \cos nx / n^3 \\ u''' = 0 & \end{array}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - (-2(\pi-x))\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right) - 0\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-(\pi-x)^2\frac{\cos nx}{n} - 2(\pi-x)\frac{\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n} + 2\pi(0) + \frac{2}{n^3}\right) - \left(-\frac{\pi^2}{n} - 0 + \frac{2}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right]$$

$b_n = 0$

$$\therefore (1) \Rightarrow f(x) = \frac{\pi^2/6}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos nx + 0\right]$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

3]. Find the fourier series for $f(x) = \begin{cases} x, & 0 < x < \pi \\ 2\pi - x, & \pi < x < 2\pi \end{cases}$

soln. :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx] \rightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}(\pi^2 - 0) + \left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \left(2\pi^2 - 2\pi^2 + \frac{\pi^2}{2}\right)\right] = \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right]$$

$a_0 = \pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$$u = x,\quad u' = 1,\quad u'' = 0 \quad\Big|\quad v = \cos nx,\quad v_1 = \frac{\sin nx}{n},\quad v_2 = -\frac{\cos nx}{n^2} \quad\Big|\quad u = 2\pi - x,\quad u' = -1,\quad u'' = 0$$

$$= \frac{1}{\pi}\left[\left(x\frac{\sin nx}{n} - 1\left(-\frac{\cos nx}{n^2}\right) + 0\right)_0^{\pi} + \left((2\pi - x)\frac{\sin nx}{n} + 1\left(\frac{-\cos nx}{n^2}\right) + 0\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi} + \left((2\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right) + \left(\frac{-1}{n^2} + \frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - \frac{2}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$v = \sin nx,\quad v_1 = -\cos nx/n,\quad v_2 = -\sin nx/n^2$$

$$= \frac{1}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right) + 0\right\}_0^{\pi} + \left\{(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right) + 0\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi} + \left(-(2\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right]$$

$$b_n = 0$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[(-1)^n - 1] \cos nx + 0$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} [(-1)^n - 1] \cos nx$$

4]. find the fourier Series for $f(x) = x \sin x, (0, 2\pi)$

Soln.:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x\, dx$$

$u = x$	$v = \sin x$
$u' = 1$	$v_1 = -\cos x$
$u'' = 0$	$v_2 = -\sin x$

$$= \frac{1}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{2\pi}$$

$$= \frac{1}{\pi}[-2\pi - 0] = -2$$

$$a_0 = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x \underset{A}{\cos nx} \underset{B}{\sin x}\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}[\sin(nx+x) - \sin(nx-x)]\, dx = \frac{1}{2\pi}\left\{\int_0^{2\pi} x \sin(n+1)x\, dx - \int_0^{2\pi} x \sin(n-1)x\, dx\right\}$$

$$= \frac{1}{2\pi}$$

$u = x$	$v = \sin(n+1)x$	$v = \sin(n-1)x$
$u' = 1$	$v_1 = -\frac{\cos(n+1)x}{(n+1)}$	$v_1 = -\frac{\cos(n-1)x}{(n-1)}$
$u'' = 0$	$v_2 = -\frac{\sin(n+1)x}{(n+1)^2}$	$v_2 = \frac{-\sin(n-1)x}{(n-1)^2}$

$$= \frac{1}{2\pi}\left[\left(x\left(-\frac{\cos(n+1)x}{n+1}\right) - \left(-\frac{\sin(n+1)x}{(n+1)^2}\right)\right)_0^{2\pi} - \left(x\left(-\frac{\cos(n-1)x}{n-1}\right) - \left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right)_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left(-\frac{x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right)_0^{2\pi} + \left(-\frac{x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right)_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi}{n+1} + \frac{2\pi}{n-1}\right]$$

$\cos(n+1)2\pi = 1$

$\sin(n+1)2\pi = 0$

$\cos(n-1)2\pi = \cos(2n\pi - 2\pi)$

$\cos 2n\pi \cos 2\pi + \sin 2n\pi \sin 2\pi$

$= 1$

$\sin(n-1)2\pi = \sin(2n\pi - 2\pi)$

$\sin 2n\pi \cos 2\pi - \cos 2n\pi \sin 2\pi$

$= 0$

$$= \frac{+2\pi}{2\pi}\left[\frac{n+1+n+1}{(n+1)(n-1)}\right]$$

$$a_n = \frac{+2}{n^2-1}$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\frac{\sin 2x}{2}\,dx$$

$$= \frac{1}{2\pi}\left[\cancel{-\frac{\cos 2x}{2}}\right]_0^{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \cancel{\frac{1}{4\pi}} \quad -1$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$u = x$, $u' = 1$, $u'' = 0$ | $v = \sin 2x$, $v_1 = -\cos 2x/2$, $v_2 = -\sin 2x/4$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - \left(\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{2}\right]$$

$$a_1 = -1/2$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\ \sin nx\ dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \ \sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\ \underset{A}{\sin nx}\ \underset{B}{\sin x}\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}\left[\cos(nx - x) - \cos(nx + x)\right] dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\cos(n-1)x\, dx - \int_0^{2\pi} x \cos(n+1)x\, dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{x\,\frac{\sin(n-1)x}{n-1} - 1\left(-\frac{\cos(n-1)x}{(n-1)^2}\right)\right\}_0^{2\pi} - \left\{x\,\frac{\sin(n+1)x}{n+1} - 1\left(-\frac{\cos(n+1)x}{(n+1)^2}\right)\right\}_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{x\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right)_0^{2\pi} - \left(\frac{x\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2}\right)_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{1}{(n-1)^2} - \frac{1}{(n-1)^2}\right) - \left(\frac{1}{(n+1)^2} - \frac{1}{(n+1)^2}\right)\right]$$

$$b_n = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\ \sin x\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\, dx - \int_0^{2\pi} \frac{x\cos 2x}{2}\, dx\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{x^2}{2}\right)_0^{2\pi} - \frac{1}{2}\left\{x\,\frac{\sin 2x}{2} - 1\left(-\frac{\cos 2x}{4}\right)\right\}_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - \frac{1}{2}\left\{\frac{1}{4} - \frac{1}{4}\right\}\right]$$

$$b_1 = \pi$$

$$f(x) = \frac{-2}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + \pi \sin x + 0$$

$$= -1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos nx + \pi \sin x$$

H.W. 1). $f(x) = \begin{cases} x \\ \pi - x \end{cases}$

3). $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$

4). $f(x) = e^{-x}, \; 0 < x < 2\pi$

Method of variation of Parameters

The second order linear differential eqn. is

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q = X$ where X is a fn. of x.

$CF = c_1 f_1 + c_2 f_2$, where c_1, c_2 are constants
f_1, f_2 are functions of x.

$PI = Pf_1 + Q f_2$

where $P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx$

$Q = \int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx$

$\int \tan x\,dx = +\log(\sec x)$

$\int \cot x\,dx = \log(\sin x)$

$\int \operatorname{cosec} x\,dx = -\log[\csc x + \cot x]$

$\int \sec x\,dx = \log(\sec x + \tan x)$

II. Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ using method of variation of parameters.

Soln.

Given $(D^2+4)y = 4\tan 2x$ where $X = 4\tan 2x$

AE

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

$CF = c_1 \cos 2x + c_2 \sin 2x$

$PI = Pf_1 + Q f_2$

Here $f_1 = \cos 2x$ | $f_2 = \sin 2x$

$f_1' = -2\sin 2x$ | $f_2' = 2\cos 2x$

Now $W = f_1 f_2' - f_2 f_1'$

$= \cos 2x[2\cos 2x] - \sin 2x(-2\sin 2x)$

$= 2\cos^2 2x + 2\sin^2 2x$

$= 2[\cos^2 2x + \sin^2 2x]$

$= 2(1) = 2$

$$P = -\int \frac{f_2 X}{W}\,dx$$

$$= -\int \frac{\sin 2x \; 4 \tan 2x}{2}\,dx$$

$$= -2\int \sin 2x \; \frac{\sin 2x}{\cos 2x}\,dx$$

$$= -2\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$= -2\int \frac{1-\cos^2 2x}{\cos 2x}\,dx = -2\left[\int \frac{1}{\cos 2x}\,dx - \int \cos 2x\,dx\right]$$

$$= -2\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right]$$

$$= -2\left[\frac{\log(\sec 2x + \tan 2x)}{2} + \frac{\sin 2x}{2}\right]$$

$$P = -\log(\sec 2x + \tan 2x) + \sin 2x$$

$$Q = \int \frac{f_1 X}{W}\,dx$$

$$= \int \frac{\cos 2x \; 4 \tan 2x}{2}\,dx$$

$$= 2\int \cos 2x \, \frac{\sin 2x}{\cos 2x}\,dx$$

$$= 2\int \sin 2x\,dx = 2\left[-\frac{\cos 2x}{2}\right]$$

$$Q = -\cos 2x$$

$$PI = Pf_1 + Qf_2$$

$$= \left[-\log(\sec 2x + \tan 2x) + \sin 2x\right]\cos 2x$$

$$- \cos 2x \sin 2x$$

$$PI = -\log(\sec 2x + \tan 2x)\cos 2x$$

$$\therefore y = CF + PI = c_1 \cos 2x + c_2 \sin 2x - \log(\sec 2x + \tan 2x)\cos 2x$$