Fourier Sine Transform:

The fourier sine transform of $f(x)$ is defined by,

$$F_S[S] = F_S[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

The inverse fourier sine transform of $F_S(S)$ is given by,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_S(S)\sin sx\,ds$$

Fourier cosine Transform:

The Fourier cosine transform of $f(x)$ is defined by

$$F_C[S] = F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

The inverse fourier cosine transform of $F_C(S)$ is given by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C[S]\cos sx\,ds$$

Parseval's Identity:

Sine Transform:

If F(S) is the fourier transform of $f(x)$, then

$$\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_S(S)]^2 ds$$

Cosine Transform:

If F(S) is the fourier transform of $f(x)$, then

$$\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_C(S)]^2 ds.$$

1]. Find the FST of $f(x)$ defined as

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Soln.:

$$F_S(S) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin Sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin Sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos Sx}{S} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos S}{S} \right]$$

2]. Find the FST of $1/x$.

Soln.: $F_S(S) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin Sx \, dx$

Put $\theta = sx \Rightarrow d\theta = s\,dx$

$$\frac{d\theta}{s} = dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta}\, d\theta$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} \qquad \because \int_0^\infty \frac{\sin\theta}{\theta}\, d\theta = \pi/2$$

$$= \sqrt{\frac{\pi}{2}}$$

3]. Find the FCT of $2e^{-3x} + 3e^{-2x}$

Soln. :

$$F_c(S) = \sqrt{\frac{2}{\pi}} \int_0^\infty (2e^{-3x} + 3e^{-2x}) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[2\int_0^\infty e^{-3x} \cos sx\, dx + 3\int_0^\infty e^{-2x} \cos sx\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[2\frac{+3}{s^2+3^2} + 3\frac{2}{s^2+2^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{6}{s^2+9} + \frac{6}{s^2+4} \right]$$

4]. Find the ~~FCT~~ FST of e^{-ax} and deduce that using I & P.

Soln. :

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

5]. Find the FCT of $\frac{e^{-ax}}{x}$ and hence, find $F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$

Soln. :

$$F_c[s] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx$$

$$\frac{d}{ds}F_c[s] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\cos sx\right)dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}(-x\sin sx)\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\frac{d}{ds}F_c[s] = -\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2} \qquad \because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

~~$F_c(s)$~~

Integrating, we get

$$F_c[s] = -\sqrt{\frac{2}{\pi}}\int \frac{s}{s^2+a^2}\,ds$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int \frac{2s}{s^2+a^2}\,ds$$

$$F_c\left[\frac{e^{-ax}}{x}\right] = -\frac{1}{\sqrt{2\pi}}\log(s^2+a^2)$$

6. Find the FST of $\frac{e^{-ax}}{x}$ and hence find $F_S\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$

Soln.

$$F_S\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$$

differentiate w.r. to 's'

$$\frac{d}{ds}F_S\left[\frac{e^{-ax}}{x}\right] = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial s}(\sin sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}(x\cos sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$$

Integrating w.r. to s, we get

$$F_S\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\,a\int\frac{1}{s^2+a^2}\,ds$$

$$= \sqrt{\frac{2}{\pi}}\,a\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$$

$$F_S\left[\frac{e^{-bx}}{x}\right] = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)$$

7. Find FST and FCT of $e^{-a|x|}$. Hence Show that i) $\int_0^\infty \frac{\cos sx}{x^2+a^2}\,dx = \frac{\pi}{2a} e^{-as}$

ii) $\int_0^\infty \frac{x \sin sx}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-as}$

Soln.

$$F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

Now $F_S[e^{-a|x|}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$$

Taking Inverse FST

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S[s] \sin sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2+a^2} \, ds$$

Replace s by x and x by s,

$$\int_0^\infty \frac{x \sin xs}{x^2+a^2} \, dx = \frac{\pi}{2} e^{-as}$$

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_C[e^{-a|x|}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

Taking Inverse FCT,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \cos sx \, ds$$

$$e^{-ax} = \frac{2}{\pi} a \int_0^\infty \frac{\cos sx}{s^2+a^2} \, ds.$$

Replace s by x and x by s,

$$\int_0^\infty \frac{\cos xs}{x^2+a^2} \, dx = \frac{\pi}{2a} e^{-as}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

Taking Inverse FCT,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \cos sx \, ds$$

$$e^{-ax} = \frac{2}{\pi} a \int_0^\infty \frac{\cos sx}{s^2+a^2} \, ds.$$

Replace s by x and x by s,

$$\int_0^\infty \frac{\cos xs}{x^2+a^2} \, dx = \frac{\pi}{2a} e^{-as}$$