Initial value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and $L[f(t)] = F(s)$ then

$$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[sF(s)]$$

Proof:

We know that

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$= sF(s) - f(0)$$

$$sF(s) = L[f'(t)] + f(0)$$

$$sF(s) = \int_0^\infty e^{-st} f'(t)\,dt + f(0)$$

Taking limit as $s\to\infty$ on both sides, we get

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left\{\int_0^\infty e^{-st} f'(t)\,dt + f(0)\right\}$$

$$= \lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\,dt + f(0)$$

$$= \int_0^\infty \lim_{s\to\infty} e^{-st} f'(t)\,dt + f(0)$$

$$= 0 + f(0)$$

$$= \lim_{t\to 0} f(t)$$

Hence, $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Problems:

1) Verify the initial and final value theorem for

$f(t) = 1 + e^{-t}(\sin t + \cos t)$.

Sol:- $F(s) = L\left[1 + e^{-t}\sin t + e^{-t}\cos t\right]$

$= L(1) + L(\sin t)_{s \to s+1} + L(\cos t)_{s \to s+1}$

$= \frac{1}{s} + \left(\frac{1}{s^2+1}\right)_{s \to s+1} + \left(\frac{s}{s^2+1}\right)_{s \to s+1}$

$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$

$= \frac{1}{s} + \frac{s+2}{s^2+2s+2}$

$sF(s) = s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$

Initial value theorem: $\lim_{t \to 0} f(t) = \lim_{s \to \infty} s.F(s)$

$\lim_{t \to 0} f(t) = \lim_{t \to 0}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1 + 1 = 2$

$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s\left(\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right)$

$= \lim_{s \to \infty} 1 + \frac{s^2+2s}{s^2+2s\left(1+\frac{2}{s^2+2s}\right)}$ $\quad \lim \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{2}{s}\right)\left(1+\frac{2}{s^2+2s}\right)}$

$= 1 + 1$

$= 2$

Hence $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) = 2$

Initial value theorem is verified.

Final Value theorem: $\underset{t\to\infty}{lt} f(t) = \underset{s\to 0}{lt} sF(s)$

$$\underset{t\to\infty}{lt} f(t) = \underset{t\to\infty}{lt} \left[1 + e^{-t}(\sin t + \cos t)\right] = 1$$

$$\underset{s\to 0}{lt} sF(s) = \underset{s\to 0}{lt} s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= 1$$

Hence $\underset{t\to\infty}{lt} f(t) = \underset{s\to 0}{lt} sF(s) = 1$

$\therefore$ Final value theorem is verified.

Laplace transform of some Special functions:

Unit Step function:

The unit step function also called Heavisides unit function is defined as,

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

This is the unit step functions at $t=a$. It can also be denoted by $H(t-a)$ or $u_a(t)$.

Result:

Laplace Transform of unit step function is $\frac{e^{-as}}{s}$.

ie,) $L\left[u(t-a)\right] = \frac{e^{-as}}{s}$.

Proof:

$$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$L[u(t-a)] = \frac{e^{-as}}{s} \quad (s>0)$$

Transforms of Periodic functions:

A function $f(x)$ is said to be Periodic if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace transformation of a periodic function $f(t)$ with Period p given by,

$$\boxed{L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt}$$

Problems:

1) Find the Laplace transform of the rectangular wave given by $f(t)=\begin{cases}1 & , 0<t<b \\ -1 & , b<t<2b\end{cases}$.

Sol: Given: $f(t)=\begin{cases}1 & , 0<t<b \\ -1 & , b<t<2b\end{cases}$

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt .$$

This function is Periodic in the interval $(0, 2b)$ with Period $2b$.

$$L[f(t)] = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st}\,dt + \int_b^{2b} e^{-st}(-1)\,dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\left(\frac{e^{-st}}{-s}\right)_0^b - \left(\frac{e^{-st}}{-s}\right)_b^{2b} \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{-1}{s}\left(e^{-st}\right)_0^b + \frac{1}{s}\left(e^{-st}\right)_b^{2b} \right]$$

$$= \frac{1}{s(1-e^{-2bs})} \left[-(e^{-bs}-1) + (e^{-2bs} - e^{-bs}) \right]$$

$$= \frac{-e^{-bs}+1+(e^{-bs})^2 - e^{-bs}}{s(1-e^{-2bs})}$$

$$= \frac{1-2e^{-bs}+e^{-bs}}{s(1-e^{-2bs})}$$

$$= \frac{1}{s(1-e^{-bs})(1+e^{-bs})}(1-e^{-bs})^2$$

$$= \frac{1}{s}\left(\frac{1-e^{-bs}}{1+e^{-bs}}\right)$$

$$= \frac{1}{s}\left(\frac{e^{sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}\right)$$

$$= \frac{1}{s}\tanh\left(\frac{bs}{2}\right).$$

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2) Find the Laplace transform of the half wave rectifier function $f(t) = \begin{cases} \sin\omega t, & 0<t<\frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega}<t<\frac{2\pi}{\omega} \end{cases}$

Sol: $$L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + 0\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t-\omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-s\pi/\omega}\cdot\omega+\omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left(\cancel{1+e^{-s\pi/\omega}}\right)}{\left(1-e^{-s\pi/\omega}\right)\left(\cancel{1+e^{-s\pi/\omega}}\right)\left(s^2+\omega^2\right)}$$

$$= \frac{\omega}{\left(1-e^{-s\pi/\omega}\right)\left(s^2+\omega^2\right)}$$

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3) Find the laplace transform of

$$f(t) = \begin{cases} t, & 0<t<a \\ 2a-t, & a<t<2a \end{cases} \quad \text{with } f(t+2a) = f(t).$$

Sol:- $L[f(t)] = \dfrac{1}{1-e^{-2as}} \displaystyle\int_0^{2a} e^{-st} f(t)\,dt.$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\, t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[-t\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^a + \left[-(2a-t)\frac{e^{-st}}{s} + \frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\left(-a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2}\right) - \left(\frac{-1}{s^2}\right)\right] + \left[\left(\frac{e^{-2as}}{s^2}\right) - \left(\frac{-ae^{-as}}{s} + \frac{e^{-as}}{s^2}\right)\right]\right\}$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as}-2e^{-as}}{s^2}\right]$$

$$= \frac{(1-e^{-as})^2}{s^2(1+e^{-as})\cancel{(1-e^{-as})}}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$