UNIT IV COMPLEX INTEGRATION

Part-A

Problem 1 Evaluate $\int_C \frac{z}{(z-1)^3}\,dz$ where C is $|z|=2$ using Cauchy's integral formula

Solution:

Given $\int_C \frac{z}{(z-1)^3}\,dz$

Here $f(z)=z$, $a=1$ lies inside $|z|=2$

$$\therefore \int_C \frac{zdz}{(z-1)^3}=\frac{2\pi i}{2!}f''(1)$$

$$=\pi i[0] \because f''(1)=0$$

$$\therefore \int_C \frac{zdz}{(z-1)^3}=0\,.$$

Problem 2 State Cauchy's Integral formula

Solution:

If $f(z)$ is analytic inside and on a closed curve C that encloses a simply connected region R and if 'a' is any point in R, then $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$.

Problem 3 Evaluate $\int_C e^{\frac{1}{z}}dz$ where C is $|z-2|=1$.

Solution:

$e^{\frac{1}{z}}$ is analytic inside and on C.

Hence by Cauchy's integral theorem $\int_C e^{\frac{1}{z}}dz=0$

Problem 1 Using Cauchy integral formula evaluate $\int_C \frac{dz}{(z+1)^2(z-2)}$ where C the circle $|z| = \frac{3}{2}$.

Solution:

Here $z = -1$ is a pole lies inside the circle

$z = 2$ is a pole lies out side the circle

$$\therefore \int_C \frac{dz}{(z+1)^2(z-2)} = \int \frac{\frac{1}{z-2}}{(z+1)^2}\, dz$$

Here $f(z) = \frac{1}{z-2}$

$$f'(z) = -\frac{1}{(z-2)^2}$$

Hence by Cauchy integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

$$\int_C \frac{dz}{(z+1)^2(z-2)} = \int_C \frac{\frac{1}{z-2}}{[z-(-1)]^2} dz$$

$$= \frac{2\pi i}{1!} f'(-1)$$

$$= 2\pi i \left[\frac{-1}{(-1-2)^2}\right] \quad \left(\therefore f'|z| = \frac{-1}{(z-2)^2}\right)$$

$$= 2\pi i \left[\frac{-1}{9}\right]$$

$$= \frac{-2}{9}\pi i.$$

Problem 2 Evaluate $\int_C \frac{z-2}{z(z-1)} dz$ where C is the circle $|z| = 3$.

Solution:

W.K.T $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$

Given $\int_C \frac{z-2}{z(z-1)} dz$ Here $z = 0$, $z = 1$ lies inside the circle

Also $f(z) = z - 2$

Now $\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$

Put $z = 0 \Rightarrow A = -1$

$z = 1 \Rightarrow B = 1$

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$\int_C \frac{z-2}{z(z-1)} dz = -\int_C \frac{z-2}{z} dz + \int_C \frac{z-2}{z-1} dz$$

$$= -2\pi i\ f(0) + 2\pi i\ f(1)$$

$$= 2\pi i[f(1) - f(0)]$$

$$= 2\pi i[-1-(-2)]$$

$$= 2\pi i[2-1] = 2\pi i.$$