Problem 7 Expand $f(z)=\sin z$ into a Taylor's series about $z=\frac{\pi}{4}$.

Solution:

Given $f(z)=\sin z$

$f'(z)=\cos z$

$f''(z)=-\sin z$

$f'''(z)=-\cos z$

Here $a=\frac{\pi}{4}$

$\therefore f\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$f'\left(\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$f''\left(\frac{\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

$f'''\left(\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

W.K.T Taylor's series of $f(z)$ at $z=a$ is

$$f(z)=f(a)+\frac{z-a}{1!}f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots$$

$$f(z)=f\left(\frac{\pi}{4}\right)+\frac{z-\frac{\pi}{4}}{1!}f'\left(\frac{\pi}{4}\right)+\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\ldots$$

$$=\frac{1}{\sqrt{2}}+\left(z-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}-\left(\frac{z-\frac{\pi}{4}}{2}\right)^2\left(\frac{1}{\sqrt{2}}\right)+\ldots$$

Problem 5 Expand $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ as a Taylor series valid in the region $|z|<2$.

Solution:

Given $f(z)=\frac{z^2-1}{(z+2)(z+3)}$

Now $(z+2)(z+3)=z^2+5z+6$

$$\therefore \frac{z^2-1}{(z+2)(z+3)}=1+\frac{-5z-7}{(z+2)(z+3)}$$

Now $\frac{-5z-7}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$

$-5z-7=A(z+3)+B(z+2)$

Put $z=-2$

$A=3$

Put $z=-3$

$B=-8$

$$\therefore f(z)=1+\frac{3}{z+2}-\frac{8}{z+3}$$

Given $|z|<2$

$$f(z)=1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{2}\left(1-\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2+\dots\right)$$

$$=1+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n-\frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$f(z)=1+\sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}}-\frac{8}{3^{n+1}}\right]z^n .$$