Problem 8 Evaluate $\int_C \frac{z\sec z}{(1-z^2)}dz$ where C is the ellipse $4x^2+9y^2=9$, using Cauchy's residue theorem.

Solution:

Equation of ellipse is

$4x^2+9y^2=9$

$\frac{x^2}{9/4}+\frac{y^2}{1}=1$

i.e., $\frac{x^2}{\left(3/2\right)^2}+\frac{y^2}{1}=1$

$\therefore$ Major axis is $\frac{3}{2}$, Minor axis is 1.

The ellipse meets the x axis at $\pm\frac{3}{2}$ and the y axis at ±1

Given $f(z)=\frac{z\sec z}{1-z^2}$

$=\frac{z}{(1+z)(1-z)\cos z}$

The poles are the solutions of $(1+z)(1-z)\cos z=0$

i.e., $z=-1,\ z=1$ are simple poles and $z=(2n+1)\frac{\pi}{2}$

Out of these poles $z\pm1$ lies inside the ellipse

$z=\pm\frac{\pi}{4},\ \pm3\frac{\pi}{4}$ lies outside the ellipse

$\left[\operatorname{Res} f(z)\right]_{z=1}=\underset{z\to1}{Lt}(z-1)\frac{z}{(1+z)(1-z)\cos z}$

$=\underset{z\to1}{Lt}\frac{-z}{(1+z)\cos z}=\frac{-1}{2\cos1}$

$\left[\operatorname{Res} f(z)\right]_{z=-1}=\underset{z\to-1}{Lt}(z+1)\frac{z}{(1+z)(1-z)\cos z}$

$=\underset{z\to-1}{Lt}\frac{z}{(1-z)\cos z}$

$$= \frac{-1}{2\cos 1} = \frac{-1}{2\cos 1}$$

$$\therefore \int_C \frac{z\sec z}{1-z^2}\,dz = 2\pi i\left[\text{sum of the residues }\right]$$

$$= 2\pi i\left[\frac{-1}{2\cos 1} - \frac{1}{2\cos 1}\right]$$

$$= -2\pi i\left[\sec 1\right].$$

Problem 9 Using Cauchy integral formula evaluate (i) $\int_C \frac{z+4}{z^2+2z+5}\,dz$, where C is the circle $|z+1-i| = 2$ (ii) $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$, C is the circle $|z| = \frac{3}{2}$.

Solution:

(i) Given $|z+1-i| = 2$

$|z-(-1+i)| = 2$ is a circle whose centre is $-1+i$ and radius 2.

i.e., centre $(-1,1)$ and radius 2

$z^2+2z+5 = \left[z-(-1+2i)\right]\left[z-(-1-2i\right]$

$-1+2i$ i.e., $(-1,2)$ lies inside the C

$-1-2i$ i.e., $(-1,-2)$ lies out side the C

$$\left[\therefore z^2+2z+5 = 0 \Rightarrow z = -2 \pm \sqrt{\frac{4-20}{2}},\ z = -1 \pm 2i\right]$$

$$\therefore \int_C \frac{z+4}{\left[z-(-1+2i)\right]\left[z-(-1-2i)\right]}\,dz$$

$$= \int_C \frac{\frac{z+4}{\left[z-(-1-2i)\right]}}{z-(-1+2i)}\,dz$$

Hence $f(z) = \frac{z+4}{\left[z-(-1-2i)\right]}$

Here by Cauchy integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{z+4}{z^2+2z+5} = 2\pi i\, f(-1+2i)$$

$$= 2\pi i\left[\frac{-1+2i+4}{(-1+2i)-(-1-2i)}\right]$$

$$= 2\pi i\left[\frac{3+2i}{4i}\right] = \frac{\pi}{2}[3+2i].$$

(ii) $\int_C \frac{4-3z}{z(z-1)(z-2)}dz$

$z=0,\ z=1$ lie inside the circle $|z| = \frac{3}{2}$

$z=2$ lies outside the circle

$\therefore \frac{4-3z}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$

$4-3z = A(z-1)(z-2) + B(z)(z-2) + C(z)(z-1)$

Put $z=0$

$4 = 4A$

$A = 1$

Put $z=1$

$B = -1$

Put $z=2$

$C = -1$

$\therefore \frac{4-3z}{z(z-1)(z-2)} = \frac{2}{z} - \frac{1}{z-1} - \frac{1}{z-2}$

$\int_C \frac{4-3z}{z(z-1)(z-2)}dz = \int_C \frac{2}{z}\,dz - \int_C \frac{1}{z-1}dz - \int_C \frac{1}{z-2}dz$

$\int_C \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(0)$

$$= 2\,[2\pi i\, f(0)] - 2\pi i\ f(1) - 0$$

$$= 4\pi i\ f(0) - 2\pi i\ f(1)$$

$$= 4\pi i\ (1) - 2\pi i\ (1)$$

$$= 2\pi i \qquad (\therefore f(0) = 1\, f(1) = 1)$$