Initial value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and

$L[f(t)] = F(s)$ then

$$\lim_{t \to 0} [f(t)] = \lim_{s \to \infty} [sF(s)]$$

Proof:

We know that

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$= sF(s) - f(0)$$

$$sF(s) = L[f'(t)] + f(0)$$

$$sF(s) = \int_0^\infty e^{-st} f'(t)\,dt + f(0)$$

Taking limit as $s \to \infty$ on both sides, we get

$$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \left\{ \int_0^\infty e^{-st} f'(t)\,dt + f(0) \right\}$$

$$= \lim_{s \to \infty} \int_0^\infty e^{-st} f'(t)\,dt + f(0)$$

$$= \int_0^\infty \lim_{s \to \infty} e^{-st} f'(t)\,dt + f(0)$$

$$= 0 + f(0)$$

$$= \lim_{t \to 0} f(t)$$

Hence, $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$.

Final value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and

$$L[f(t)] = \lim_{s\to 0}[sF(s)]$$

Proof:

We know that

$$L[f'(t)] = sL[f(t)] - f(0)$$
$$= sF(s) - f(0)$$
$$sF(s) = L[f'(t)] + f(0)$$

Taking limit $s\to 0$ on both sides, we get

$$\lim_{s\to 0}[sF(s)] = \lim_{s\to 0}\left\{\int_0^\infty e^{-st} f'(t)\,dt + f(0)\right\}$$
$$= \int_0^\infty \lim_{s\to 0} e^{-st} f'(t) + f(0)$$
$$= \int_0^\infty f'(t)\,dt + f(0)$$
$$= [f(t)]_0^\infty + f(0)$$
$$= f(\infty) - f(0) + f(0)$$
$$= \lim_{t\to\infty} f(t)$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$$

Problems:

1) Verify the initial and final value theorem for

$f(t) = 1 + e^{-t}(\sin t + \cos t)$.

Sol:- $F(s) = L\left[1 + e^{-t}\sin t + e^{-t}\cos t\right]$

$= L(1) + L(\sin t)_{s \to s+1} + L(\cos t)_{s \to s+1}$

$= \frac{1}{s} + \left(\frac{1}{s^2+1}\right)_{s \to s+1} + \left(\frac{s}{s^2+1}\right)_{s \to s+1}$

$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$

$= \frac{1}{s} + \frac{s+2}{s^2+2s+2}$

$sF(s) = s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$

Initial Value theorem: $\lim_{t \to 0} f(t) = \lim_{s \to \infty} s F(s)$

$\lim_{t \to 0} f(t) = \lim_{t \to 0} \left[1 + e^{-t}(\sin t + \cos t)\right] = 1 + 1 = 2$

$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s\left(\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right)$

$= \lim_{s \to \infty} 1 + \frac{s^2+2s}{s^2+2s\left(1+\frac{2}{s^2+2s}\right)}$ $\quad 1 + \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{2}{s}\right)\left(1+\frac{2}{s^2+2s}\right)}$

$= 1 + 1$

$= 2$

Hence $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) = 2$

Initial value theorem is verified.

Final Value theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \left[1+e^{-t}(\sin t+\cos t)\right] = 1$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left[\frac{1}{s}+\frac{s+2}{s^2+2s+2}\right]$$

$$= 1$$

Hence $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 1$

$\therefore$ Final value theorem is verified.

Laplace transform of some Special functions:

Unit Step function:

The unit step function also called Heavisides unit function is defined as,

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$$

This is the unit step functions at $t=a$. It can also be denoted by $H(t-a)$ or $u_a(t)$.

Result:

Laplace Transform of unit step function is $\frac{e^{-as}}{s}$.

i.e.) $L[u(t-a)] = \frac{e^{-as}}{s}$