4

FAILURE THEORIES

Ductile and brittle materials - Maximum principal stress theory - Maximum principal strain theory - Maximum shear stress theory - distortion energy theory - Octahedral shear stress theory.

Due to the large number of examples of compound stresses met with in engineering practise, the failure causes set under such conditions has attracted considerable attention. In every component and at every point, the compound stresses can be evaluated by different theories. Certain theories have been advanced to explain the cause of failure and many of theories have received considerable experimental investigation.

The principal theories are,

1) Maximum Principal stress theory

2) Maximum Principal strain theory

3) Maximum Shear stress theory or Stress difference theory

4) Strain energy theory

5) Shear strain energy theory or Distortion energy theory

6) Octahedral Shear stress theory

4.1 MAXIMUM PRINCIPAL STRESS THEORY

Maximum Principal stress theory is the simplest and the oldest theory of failure. According to this theory, "failure will occur when the maximum principal tensile stress (σ_1) in the complex system reaches the value of the maximum stress at the elastic limit (σ_{et}) in simple tension or the minimum principal stress (that is, the maximum principal compressive stress) reaches the elastic limit stress (σ_{ec}) in simple compression".

i.e. $\sigma_1 = \sigma_{et}$ (In simple tension)

$|\sigma_3| = \sigma_{ec}$ (In simple compression) ... 4.1

If the maximum principal stress is the design criterion, the maximum principal stress must not exceed the working stress for the material. Hence,

$$\sigma_1 = \sigma_3 \quad \text{... 4.2}$$

This theory disregards the effect of other stresses on other planes throughout the element. For brittle materials which do not fail by yielding but fail by brittle fracture, the maximum principal stress theory is considered to be reasonably satisfactory.

This theory is only valid for ordinary cast-irons and brittle metals.

The maximum principal stress theory is contradicted in the following cases:

- On a mild steel specimen when simple tension test is carried out sliding occurs approximately 45° to the axis of the specimen, this failure is due to the maximum shear stress rather than the direct stress.
- It has been found that a material which is even though weak in simple compression, yet it can sustain hydrostatic pressure far in excess of the elastic limit in simple compression.

4.2 MAXIMUM PRINCIPAL STRAIN THEORY

This theory states that, "the failure of a material occurs when the principal tensile strain in the material reaches the strain at the elastic limit in simple tension or when the minimum principal strain (i.e. maximum principal compressive strain) reaches the elastic limit strain in simple compression".

Principal strain in the direction of principal stress σ_1,

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right] \quad \text{... 4.3}$$

Principal strain in the direction of principal stress σ_3,

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right] \quad \text{... 4.4}$$

The conditions to cause failure according to the maximum principal strain theory,

$$\varepsilon_1 > \frac{\sigma_{et}}{E} \qquad (\varepsilon_1 \rightarrow \text{tension})$$

and

$$|\varepsilon_3| > \frac{\sigma_{ec}}{E} \qquad (\varepsilon_3 \rightarrow \text{compression})$$

From equations 4.3 and 4.4

$$\therefore \frac{1}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right] > \frac{\sigma_{et}}{E}$$

and

$$\frac{1}{E}\left[\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right] > \frac{\sigma_{ec}}{E}$$

The above equations are reduced to,

$$\sigma_1 - \nu(\sigma_2 + \sigma_3) > \sigma_{et} \text{ and}$$

$$\sigma_3 - \nu(\sigma_1 + \sigma_2) > \sigma_{ec} \quad \text{... 4.5}$$

From equation 4.5 we conclude to prevent the failure,

$$\sigma_1 - \nu(\sigma_2 + \sigma_3) < \sigma_{et}$$

$$\sigma_3 - \nu(\sigma_1 + \sigma_2) < \sigma_{ec} \quad \text{... 4.6}$$

The equation for the point of elastic fracture is obtained as,

$$\sigma_1 - \nu(\sigma_2 + \sigma_3) = \sigma_{et}$$

$$|\sigma_3 - \nu(\sigma_1 + \sigma_2)| = \sigma_{ec} \quad \text{... 4.7}$$

Also for the design purposes,

$$\sigma_3 - \nu(\sigma_1 + \sigma_2) = \sigma_t$$

$$|\sigma_3 - \nu(\sigma_1 + \sigma_2)| = \sigma_c \quad \text{... 4.8}$$

where, σ_t and σ_c are the safe stresses.

This theory is contradicted in the following cases:

- The theory overestimates the behaviour of ductile materials.
- The theory does not fit well with the experimental results except for brittle materials by biaxial tension - compression state of stress and is not much used in practice.

4.3 MAXIMUM SHEAR STRESS THEORY OR STRESS DIFFERENCE THEORY

This theory is also called *Guest's* or *Tresca's* theory.

This theory states that, "failure will occur when the maximum shear stress (τ_{max}) in the complex system reaches the value of the maximum shear stress in the simple tension at the elastic limit". i.e.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{et}}{2} \quad \text{(in simple tension)}$$

or

$$\sigma_{et} = \sigma_1 - \sigma_3 \quad \text{... 4.9}$$

The maximum stress difference is to be equated to σ_{et}. In case of two dimensional tensile stress system, third stress must be taken as zero and then the maximum stress difference calculated to equate it to σ_{et}. Also this theory has been found to give quite satisfactory results for ductile materials.

This theory is contradicted in the following cases:

- The theory does not give accurate results for the state of stress of pure shear in which the maximum amount of shear is developed (i.e. torsion test).
- The theory does not give as close results as found by experiments on ductile materials and however, it gives safe results.
- This theory is not applicable in the case, where the state of stress consists of triaxial tensile stresses of nearly equal magnitude reducing the shear stress to a small magnitude, so that failure would be, by brittle fracture rather than by yielding.

4.4 STRAIN ENERGY THEORY

This theory states that, "the failure of a material occurs when the total strain energy in the material reaches the total strain energy of the material at the elastic limit in simple tension".

In a 3-D stress system, the strain energy per unit volume is given by,

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3\right)\right] \quad \text{.. 4.10}$$

Hence at the point of failure,

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3\right)\right] = \frac{\sigma_e^2}{2E}$$

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3) = \sigma_e^2 \quad \text{... 4.11}$$

While taking 2-D cases, $\sigma_3 = 0$ then equation 4.11 reduced to,

$$\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 = \sigma_e^2 \quad \text{... 4.12}$$

If the material working stress is σ, the design criteria may be stated as follows,

$$\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 \leq \sigma^2 \quad \text{... 4.13}$$

This theory is contradicted in the following cases:

- The results of this theory are similar to the experimental results for ductile materials (i.e. the materials which fail by general yielding) for which $\sigma_{et} = \sigma_{ec}$ approximately.
- The theory does not apply to materials for which σ_{et} is quite different from σ_{ec}.
- Even though the results for ductile material are close to the experimental results but not exactly equal to it.

4.5 SHEAR STRAIN ENERGY THEORY OR DISTORTION ENERGY THEORY OR MISES - HENKY THEORY

This theory states that, "the elastic failure occurs when the shear strain energy per unit volume in the stressed material reaches a value equal to the shear strain energy per unit volume at the elastic limit in the simple tension test."

Shear strain energy per unit volume is,

$$U_s = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \quad \text{... 4.14}$$

But for the simple tension test at the elastic limit, where there is only one principal stress σ_{et} is given by,

$$U_s' = \frac{1}{12G}\left[(\sigma_{et} - 0)^2 + (0 - 0)^2 + (0 - \sigma_{et})^2\right] \quad \text{... 4.15}$$

Equating two energies, (From equation 4.14 and 4.15)

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_{et}^2 \quad \text{... 4.16}$$

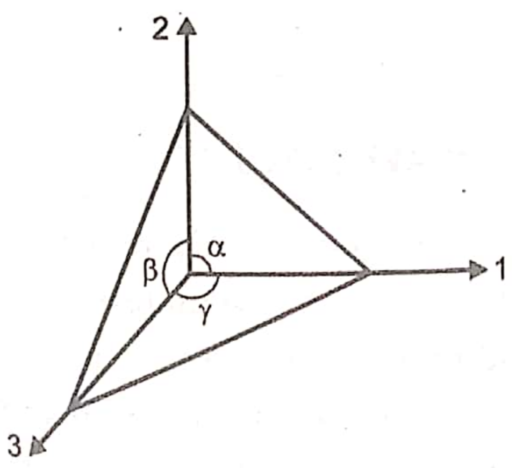
In actual design σ_{et} is replaced by safe stress σ_t in equation 4.16. The above theory has been found to give best results for ductile materials for which $\sigma_{et} = \sigma_{ec}$ approximately.

This theory is contradicted in the following cases:

- It does not agree with the experimental results for the material, for which σ_{et} is quite different from σ_{ec}.
- It gives $\sigma_{et} = 0$ for hydrostatic pressure or tension, which means that the material will never fail under any hydrostatic pressure or tension and this exactly not correct. Actually when three equal tensions are applied in three principal directions, brittle fracture occurs and as such maximum principal stress theory will give reliable results in this case.
- This theory is regarded as one to which conform most of the ductile materials under the action of various types of loading.

4.6 OCTAHEDRAL SHEAR STRESS THEORY: (VON-MISES THEORY)

This theory states that, "the failure of the material occurs when the octahedral shear stress in complex system reaches the value of maximum octahedral shear stress in simple tension at elastic limit".



$$\cos\alpha = \cos\beta = \cos\gamma = 1$$

$$l = m = n = \frac{1}{\sqrt{3}}$$

$$l^2 + m^2 + n^2 = 1$$

Figure 4.1 Octahedral Plane

Since hydrostatic stress alone does not cause yielding, we can find a material plane called the octahedral plane. The strain energy in the state of stress is determined by the octahedral shear stress is given by,

$$\tau_h = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \quad \dots 4.17$$

We expect yielding when the octahedral shear stress is equal to or exceeds a stress criterion value for failure of a given material, which is the octahedral stress criterion (τ_{ho}),

$$\tau_h \geq \tau_{ho} \quad \text{(Failure)}$$

$$\tau_h = \tau_{ho} \quad \text{(At yielding)} \quad \dots 4.18$$

For a given material under a axial load where, $\sigma_1 = \sigma_{et}$ and $\sigma_2 = \sigma_3 = 0$ while yielding occurs. So from equation 4.17 and 4.18, we get,

$$\tau_{ho} = \tau_h = \frac{1}{3}\sqrt{(\sigma_{et} - 0)^2 + (0 - 0)^2 + (0 - \sigma_{et})^2}$$

$$\tau_h = \frac{\sqrt{2}}{3}\sigma_{et} \quad \dots 4.19$$

From equation 4.17 and 4.19,

$$\frac{\sqrt{2}}{3}\sigma_{et} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\sigma_{et} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \quad .4.20$$