UNIT 4 Fourier Series and Fourier Transform

PROBLEMS UNDER FOURIER SERIES IN $(0, 2l)$

① Find the Fourier series for $f(x) = x^2$ in $0 < x < 2l$. Hence deduce

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots = \frac{\pi^2}{8}$

Solution:

$f(x)$ is defined in $(0, 2l)$

The fourier series for $f(x)$ in the interval $(0, 2l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{—} ①$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} x^2\,dx$$

$$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^{2l} = \frac{1}{l}\left[\frac{8l^3}{3}\right]$$

$$\boxed{a_0 = \frac{8l^2}{3}}$$

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$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{2l} x^2\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\cancel{x^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)} - 2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + \cancel{2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)}\right]_0^{2l}$$

$$= \frac{1}{l}\,\frac{2l^2}{n^2\pi^2}\left[x\cos\frac{n\pi x}{l}\right]_0^{2l}$$

$$= \frac{2l}{n^2\pi^2}\left(2l\cos\frac{2n\pi l}{l} - 0\cos 0\right)$$

$$= \frac{2l}{n^2\pi^2}\left(2l\cos 2n\pi\right)$$

$$\boxed{a_n = \frac{4l^2}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{2l} x^2\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[x^2\left(\frac{-\cos n\pi x/l}{\frac{n\pi}{l}}\right) - 2x\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[x^3\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^2}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[4l^2\left(\frac{-\cos 2n\pi}{\frac{n\pi}{l}}\right) + \left(\frac{2\cos 2n\pi}{\frac{n^3\pi^3}{l^2}}\right) - \left(0 + \frac{2\cos 0}{\frac{n^3\pi^3}{l^2}}\right)\right]$$

$$= \frac{1}{l}\left[\frac{-4l^2}{\frac{n\pi}{l}} + \frac{2}{\frac{n^3\pi^3}{l^3}} - \frac{2}{\frac{n^3\pi^3}{l^3}}\right]$$

$$\boxed{b_n = \frac{-4l^2}{n\pi}}$$

Substitute the values of a_0, a_n, b_n in (1), we get

$$f(x) = \frac{4l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{-4l^2}{n\pi} \sin\frac{n\pi x}{l} \quad -(2)$$

i) To deduce $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots = \frac{\pi^2}{6}$

Put $x=0$ in (2), we get

$$(2) \Rightarrow f(0) = \frac{4l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}\cos 0 - \sum_{n=1}^{\infty} \frac{4l^2}{n\pi}\sin 0$$

$$f(0) = \frac{4l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \quad -(3)$$

To find $f(0)$:

When have $f(x) = x^2$ in $0 < x < 2l$.

$x=0$ is the end of $(0,2l)$

$$f(0)=0$$
$$f(2l)=(2l)^2=4l^2$$
$$f(0)\neq f(2l)$$

Here $x=0$ is a point of discontinuity

$$\therefore f(0)=\frac{f(0)+f(2l)}{2}$$
$$=\frac{0+4l^2}{2}$$
$$f(0)=2l^2$$

Substitute the value of $f(0)$ in ③ we get

$$2l^2=\frac{4l^2}{3}+\sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}$$
$$2l^2-\frac{4l^2}{3}=\sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}$$
$$\sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}=\frac{6l^2-4l^2}{3}$$
$$\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{2l^2}{3}$$
$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$
$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6} \quad —\ (a)$$

ii) To deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

Put $x = l$ in (2), we get

$$f(l) = \frac{4l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cos\frac{n\pi l}{l} - \sum_{n=1}^{\infty} \frac{4l^2}{n\pi} \sin\frac{n\pi l}{l}$$

$$f(l) = \frac{4l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2} - 0$$

$$f(l) = \frac{4l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2} \quad \text{--- (4)}$$

To find $f(l)$:

$x = l$ lies inside $(0, 2l)$. Hence $x = l$ is a point of continuity

$\therefore f(l) = l^2$

Substitute the value of $f(l)$ in (4) we get

$$l^2 = \frac{4l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2}$$

$$\sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2} = l^2 - \frac{4l^2}{3}$$

$$\frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n = \frac{-l^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n = \frac{-\pi^2}{12}.$$

$$\frac{(-1)^1}{1^2} + \frac{(-1)^2}{2^2} + \frac{(-1)^3}{3^2} + \cdots = \frac{-\pi^2}{12}.$$

$$-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots = -\frac{\pi^2}{12}$$

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots = \frac{\pi^2}{12} \quad \text{— (b)}$$

iii) To deduce : $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$

Add (a) and (b) we get,

$$\frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{3^2}\cdots = \frac{\pi^2}{6}+\frac{\pi^2}{12}$$

$$2\left(\frac{1}{1^2}\right)+2\left(\frac{1}{3^2}\right)+\cdots = \frac{18\pi^2}{72}$$

Ans: $\boxed{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}}$

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(2) Find the Fourier series for

$$f(x) = \begin{cases} l-x, & 0<x<l \\ 0, & l<x<2l \end{cases}$$ Hence deduce.

i) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$

ii) $1-\frac{1}{3}+\frac{1}{5}-\cdots = \frac{\pi}{4}$

Solution:

$f(x)$ is defined in $(0, 2l)$.

The fourier series for $f(x)$ in the interval $(0, 2l)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad —(1)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\left[\int_0^{l} f(x)\,dx + \int_l^{2l} f(x)\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} (l-x)\,dx + \int_l^{2l} 0\,dx\right]$$

$$= \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^{l}$$

$$= \frac{1}{l}\left[l^2 - \frac{l^2}{2}\right]$$

$$= \frac{1}{l}\left(\frac{l^2}{2}\right)$$

$$\boxed{a_0 = \frac{l}{2}}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} f(x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} (l-x)\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} 0\right]$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{-1}{l}\left(\frac{l}{n\pi}\right)^2\left[\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-l}{n^2\pi^2}\left[\cos\frac{n\pi l}{l^2} - \cos 0\right]$$

$$= \frac{-l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \frac{l}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\boxed{a_n = \begin{cases} \dfrac{2l}{n^2\pi^2} & ; \ n = 1, 3, 5 \ldots \\ 0 & ; \ n = 2, 4, 6, \ldots \end{cases}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\int_0^l (l-x)\sin\frac{n\pi x}{l}\,dx + \int_l^{2l} 0\right]$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\cancel{\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)}\right]_0^l$$

$$= \frac{-1}{l} \cdot \frac{l}{n\pi}\left[(l-x)\left(\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{-1}{n\pi}\left[(0)\left(\cos\frac{n\pi l}{l}\right) - (l)(\cos 0)\right]$$

$$= \cancel{l}\ \frac{-1}{n\pi}[0 - l]$$

$$\boxed{b_n = \frac{l}{n\pi}}$$

Substitute the values of a_0, a_n, b_n in ①

$$f(x) = \frac{l}{4} + \sum_{n=1,3}^{\infty} \frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\frac{n\pi x}{l} \quad — ②$$

To prove: i) $\frac{1}{1^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{8}$

Put $x=0$ in ②, we get

$$f(0) = \frac{l}{4} + \sum_{n=1,3}^{\infty} \frac{2l}{n^2\pi^2} \quad — ③$$

To find $f(0)$:

$x=0$ is an end of point of $(0, 2l)$

$f(0) = l - 0 = l$

$f(2l) = 0$

$\therefore f(0) \neq f(2l)$

Here $x=0$ is a point of discontinuity,

$$\therefore f(0) = \frac{f(0) + f(2l)}{2}$$

$$f(0) = \frac{l+0}{2} = \frac{l}{2}$$

Substitute $f(0) = \frac{l}{2}$ in (3), we get

$$\frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2} \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$\frac{l}{2} - \frac{l}{4} = \frac{2l}{\pi^2} \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$\frac{l}{4} = \frac{2l}{\pi^2} \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$\frac{l}{4} \times \frac{\pi^2}{2l} = \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1,3}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}$$

ii) To deduce $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$

Put $x = \frac{l}{2}$ in (2), we get

$$f\left(\frac{l}{2}\right) = \frac{l}{4} + \sum_{n=1,3}^{\infty} \frac{2l}{n^2\pi^2} \cos \frac{n\pi}{2} + \sum_{n=1}^{\infty} \frac{l}{n\pi} \sin \frac{n\pi}{2}$$

$$f\left(\frac{l}{2}\right) = \frac{l}{4} + \sum_{n=1}^{\infty} \frac{l}{n\pi} \sin \frac{n\pi}{2} \quad \text{— (4)}$$

To find $f\left(\frac{l}{2}\right)$:

$x = \frac{l}{2}$ lies inside $(0, 2l)$

Here $x = \frac{l}{2}$ is a point of continuity

$f\left(\frac{l}{2}\right) = \frac{l}{2}$ in (4), we get

$$\frac{l}{2} = \frac{l}{4} + \sum_{n=1}^{\infty} \frac{l}{n\pi} \sin\frac{n\pi}{2}$$

$$\frac{l}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2} = \frac{l}{2} - \frac{l}{4} = \frac{l}{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2} = \frac{l}{4}\cdot\frac{\pi}{l}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2} = \frac{\pi}{4}.$$

Ans: $\boxed{1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}}$

~~Homework Sums:~~

3) Find the Fourier series for $f(x) = 2x - x^2$; $0 < x < 3$

Solution:

Given interval is $(0,3)$

General interval is $(0, 2l)$

$\therefore 2l = 3 \Rightarrow \boxed{l = \frac{3}{2}}$

The Fourier series for $f(x)$ in the interval $(0, 2l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3} \quad \text{(1)}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{2}{3}\int_0^{3} f(x)\,dx$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[\frac{2x^2}{3} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}\left[9 - \frac{27}{3} - (0)\right]$$

$$= \frac{2}{3}(9 - 9)$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 f(x)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[\cancel{(2x-x^2)\left(\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)} - (2-2x)\left(-\frac{\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right) + (-2)\cancel{\left(\frac{-\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)}\right]_0^3$$

$$= \frac{2}{3}\left(\frac{3}{2n\pi}\right)^2\left[(2-2x)\left(\cos\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{3}{2n^2\pi^2}\left[(-4)(\cos 2n\pi - (2)(\cos 0)\right]$$

$$= \frac{3}{2n^2\pi^2}\left[-4-2\right]$$

$$\boxed{a_n = \frac{-9}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)}\right) - (2-2x)\cancel{\left(\frac{-\sin\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right)} + (-2)\left(\frac{\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left(\frac{-3}{2n\pi}\right)(2x-x^2)\left(\cos\frac{2n\pi x}{3}\right) - 2\left(\frac{3}{2n\pi}\right)^3\left(\cos\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left\{\left[\left(\frac{-3}{2n\pi}\right)(-3)(\cos 2n\pi) - 2\left(\frac{27}{8n^3\pi^3}\right)(\cos 2n\pi)\right] - \left[\frac{-3}{2n\pi}(0)(\cos 0) - 2\left(\frac{27}{8n^3\pi^3}\right)(\cos 0)\right]\right\}$$

$$= \frac{2}{3}\left\{\left[\left(\frac{9}{2n\pi}\right)(1) - \left(\frac{27}{4n^3\pi^3}\right)(1)\right] - \left[0 - \frac{27}{4n^3\pi^3}(1)\right]\right\}$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi} - \cancel{\frac{27}{4n^3\pi^3}} + \cancel{\frac{27}{4n^3\pi^3}}\right]$$

$$\boxed{b_n = \frac{3}{n\pi}}$$

Substitute the values of a_0, a_n, b_n in ①

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} \frac{3}{n\pi}\sin\frac{2n\pi x}{3}}$$

Homework Sums:

(1) Find the Fourier series for $f(x)=(l-x)^2$ in $0<x<2l$ hence find $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$

Ans: $\boxed{\frac{l^2}{3}+\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}\ ;\ \frac{\pi^2}{6}}$

(2) Find the Fourier series for $f(x)=(l-x)^2$, $0<x<2l$, find $\sum_{n=1}^{\infty}\frac{1}{n^2}$

Ans: $\boxed{f(x)=\frac{l^2}{3}+\sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}}$

(3) Find the Fourier series expansion of $f(x)=2lx-x^2$, $0<x<2l$

Ans: $\boxed{f(x)=\frac{2l^2}{3}+\sum_{n=1}^{\infty}\frac{-4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}}$