ODD AND EVEN FUNCTIONS:

EVEN FUNCTION:

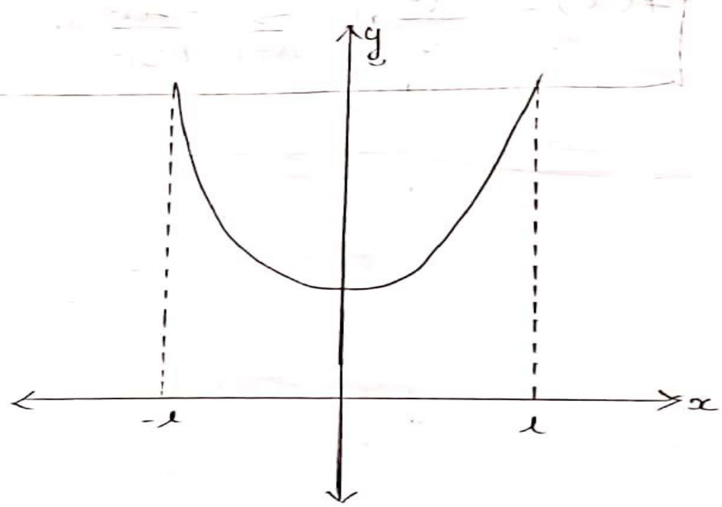
Let $f(x)$ be defined in $(-l, l)$

If $f(-x) = f(x)$, then $f(x)$ is an even function.

Note:

1. The graph of even function is symmetrical about y-axis
2. $\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$ if $f(x)$ is even.
3. Sum of two even functions is also an even function.
4. Product of two even functions is also an even function.

Graph of an Even Function:



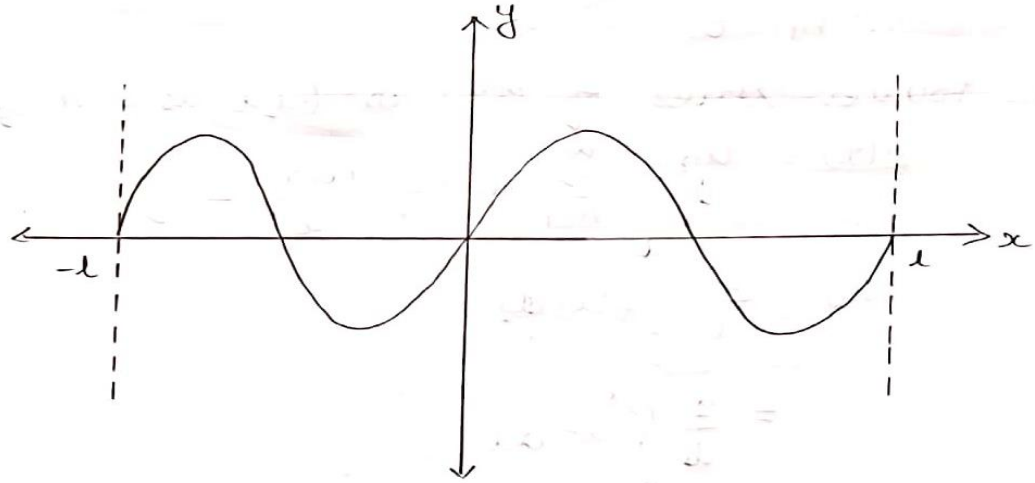
ODD FUNCTION:

Let $f(x)$ be defined in $(-l, l)$.
If $f(-x) = -f(x)$, then $f(x)$ is an odd function

Note:

1. The graph of odd function is symmetrical about origin.
2. $\int_{-l}^{l} f(x)\,dx = 0$ if $f(x)$ is odd
3. Sum of two odd function is also an odd function.
4. Product of two odd function is an even function.

Graph of an Odd function:



(3) Find the Fourier series for $f(x) = x$ in $(-l, l)$ and hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$

Solution:

$f(x)$ is defined in $(-l, l)$

In $(-l, l)$, check whether $f(x)$ is odd / even.

$$f(x) = x$$

$$f(-x) = -x = -f(x)$$

$$\therefore f(-x) = -f(x)$$

$$\therefore \boxed{a_0 = a_n = 0}$$

The Fourier series is given by

For $f(x)$ in $(-l, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{--- (1)}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \left[x \left(\frac{-\cos \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right) \right]_0^l$$

$$= \frac{-2}{l} \cdot \frac{l}{n\pi} \left[x \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-2}{n\pi}\left[l \cos n\pi - 0\right]$$

$$\boxed{b_n = \frac{-2l}{n\pi}(-1)^n}$$

Substitute the values of b_n in ①

we get

$$f(x) = \sum_{n=1}^{\infty} \frac{-2l}{n\pi}(-1)^n \sin\frac{n\pi x}{l} \quad —②$$

To deduce $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$.

Put $x = \frac{l}{2}$ in ② we get

$$f\left(\frac{l}{2}\right) = \sum_{n=1}^{\infty} \frac{-2l}{n\pi}(-1)^n \sin\frac{n\pi}{2} \quad —③$$

To find $f\left(\frac{l}{2}\right)$:

$x = \frac{l}{2}$ lies inside $(-l, l)$.

Here $x = \frac{l}{2}$ is a point of continuity

$$f\left(\frac{l}{2}\right) = \frac{l}{2}.$$

$$③ \Rightarrow \frac{l}{2} = \sum_{n=1}^{\infty} \frac{-2l}{n\pi}(-1)^n \sin\frac{n\pi}{2}.$$

$$-\frac{2l}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin\frac{n\pi}{2} = \frac{l}{2}.$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi}{2} = \frac{1}{2}\left(\frac{\pi}{-2}\right)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi}{2} = \frac{-\pi}{4}$$

$$\frac{(-1)}{1}(1) + 0 + \frac{(-1)}{3}(-1) + \cdots = \frac{-\pi}{4}$$

Ans : $\boxed{1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}}$

Homework sums:

① Find the fourier series for $f(x) = x^2$ in $(-\pi, \pi)$ and hence deduce

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

Ans: $\boxed{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx}$

② Find the Fourier series for

$$f(x) = \begin{cases} 1 + \frac{x}{\pi} ; & -\pi < x < 0 \\ 1 - \frac{x}{\pi} ; & 0 < x < \pi \end{cases}$$ and hence

deduce $\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$

Ans: $\boxed{f(x) = \frac{1}{2} + \sum_{n=1,3,5}^{\infty} \frac{4}{n^2\pi^2} \cos nx}$