PROBLEMS USING PERSEVAL'S IDENTITY IN FOURIER SINE TRANSFORM & FOURIER COSINE TRANSFORM.

$$\int_0^\infty F_S(s) G_S(s)\, ds = \int_0^\infty f(x) g(x)\, dx$$

where, $F_S(s) = F_S[f(x)]$

$G_S(s) = F_S[g(x)]$

$$\int_0^\infty F_C(s) G_C(s)\, ds = \int_0^\infty f(x) g(x)\, dx$$

where $F_C(s) = F_C[f(x)]$

$G_C(s) = F_C[g(x)]$

① Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$; $a, b > 0$

Solution:

Since the numerator in the integrant is dx, we use Fourier cosine transform. Since denominator $(x^2+a^2)(x^2+b^2)$ we take $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

$$F_C(s) = F_C[f(x)]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$F_c(s) = \boxed{\sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}}$$

Similarly

$$G_c(s) = \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2}$$

Parseval's Identity in Fourier cosine transform

$$\int_0^\infty F_c(s) G_c(s) \, ds = \int_0^\infty f(x) g(x) \, dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \sqrt{\frac{2}{\pi}} \frac{b}{s^2+b^2} \, ds = \int_0^\infty e^{-ax} e^{-bx} \, dx$$

$$\frac{2}{\pi} ab \int_0^\infty \left(\frac{1}{a^2+s^2}\right)\left(\frac{1}{s^2+b^2}\right) ds = \int_0^\infty e^{-(a+b)x} \, dx$$

$$\frac{2}{\pi} ab \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= 0 - \frac{1}{-(a+b)}$$

$$\frac{2}{\pi} ab \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a+b}$$

$$\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2} \frac{1}{ab(a+b)}$$

Put $s = x$; $ds = dx$

$$\boxed{\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}}$$

(5) Evaluate $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$; $a>0$; $b>0$

Solution:

Since the numerator of the integrant is $x^2 dx$, so we use Fourier sine transform.

Denominator is $(x^2+a^2)(x^2+b^2)$

Take $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

$$F_s(s) = F_s[f(x)]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$$

Similarly $G_s(s) = \sqrt{\frac{2}{\pi}} \frac{s}{b^2+s^2}$

Parseval's Identity in Fourier sine transforms

$$\int_0^\infty F_s(s)\, G_s(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\, \frac{s}{a^2+s^2} \sqrt{\frac{2}{\pi}}\, \frac{s}{s^2+b^2}\, ds = \int_0^\infty e^{-ax}\, e^{-bx}\, dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{s}{a^2+s^2}\, \frac{s}{s^2+b^2}\, ds = \int_0^\infty e^{-(a+b)x}\, dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^\infty$$

$$= \frac{1}{(a+b)}$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = \frac{\pi}{2(a+b)}.$$

Put $s = x;\ ds = dx$

(3) Use Parseval's Identity for Fourier cosine and sine transform for e^{-ax}, evaluate

(i) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

(ii) $\displaystyle\int_0^\infty \frac{x^2\, dx}{(x^2+a^2)^2}$.

Solution:

(i) To evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

Since numerator of the integrant is dx, use cosine transform.

Denominator is

$$(x^2+a^2)^2 = (x^2+a^2)(x^2+a^2)$$

Here $f(x) = e^{-ax}$; $g(x) = e^{-ax}$

$$F_C(s) = F_C[f(x)]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$F_C(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$G_C(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

Parseval's Idendtity in Fourier cosine transform is

$$\int_0^\infty F_C(s)\, G_C(s)\, ds = \int_0^\infty f(x) g(x) \, dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\sqrt{\frac{2}{\pi}}\frac{a}{a^2+b^2}\,ds = \int_0^\infty e^{-2ax}\,dx$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)^2} = \int_0^\infty e^{-2ax}\,dx$$

$$= \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)^2} = \frac{1}{2a}$$

$$\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{\pi}{4a^3}$$

Put $s=x$; $ds = dx$

ii) To evaluate $\int_0^\infty \frac{x^2dx}{(x^2+a^2)^2}$.

Since the numerator of the integrant is x^2dx, we use sine transform

Denominator is $(x^2+a^2)^2 = (x^2+a^2)(x^2+a^2)$

Take $f(x) = e^{-ax}$; $g(x) = e^{-ax}$

$$F_s(s) = F_s[f(x)]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$$

Similarly,

$$G_c(s) = \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$$

Parseval's Identity in Fourier sine transforms

$$\int_0^\infty F_s(s)\, G_s(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2} \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\, ds = \int_0^\infty e^{-ax} e^{-ax}\, dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{s^2}{(a^2+s^2)(a^2+s^2)}\, ds = \int_0^\infty e^{-2ax}\, dx$$

$$= \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= 0 - \left(\frac{1}{-2a}\right)$$

$$\frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds = \frac{1}{2a}.$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds = \frac{\pi}{4a}$$

Put $s = x \Rightarrow ds = dx$

$$\boxed{\int_0^\infty \frac{x^2\, dx}{(x^2+a^2)^2} = \frac{\pi}{4a}}$$