cauchy's integral theorem :-

If a function $f(z)$ is a analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C, then $\int_C f(z)\,dz = 0$.

cauchy's integral formula

If $f(z)$ is analytic inside and on simple closed curve 'c' and 'a' be any point insides C then ~~th~~ $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$, where the integration being taken around the anti-clock wise direction.

cauchy's integral formula for derivatives :-

If the function $f(z)$ is analytic inside and on a simple closed curve 'c', then and 'a' be any point inside C, then,

$$\int_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i f'(a)$$

$$\int_C \frac{f(z)}{(z-a)^3}\,dz = 2\pi i f''(a)$$

In general,

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = 2\pi i f^{(n)}(a)$$

If the point 'a' lies outside the circle or region, then $\int_C \frac{f(z)}{(z-a)}\,dz = 0$

Problem :-

1) Evaluate $\int_C \frac{\cos \pi z}{z-1}\,dz$ using cauchy's integral

formula, where c is the circle $|z| = 2$

Soln:- Here $z = 1$, $|1| = 1 < 2$ which lies inside c.

By cauchy's Integral formula,

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\int_c \frac{\cos \pi z}{z-1}\,dz = 2\pi i f(1)$$

$$= 2\pi i(-1)$$

$$\int_c \frac{\cos \pi z}{z-1}\,dz = -2\pi i$$

2) Evaluate $\int_c \frac{dz}{(z-3)^2}$ where c is the circle $|z| = 1$

Soln:-

c, $|z| = 1$

$c : |z| = 1$

Here $z = 3$, $|3| = 3 > 1$; which lies outside c,

By cauchy's Integral formula,

$$\int_c \frac{dz}{(z-3)^2} = 0$$

3) Evaluate $\int_c \frac{e^z}{z+1}\,dz$, where c is the circle $|z+\frac{1}{2}| = 1$

Soln:- Given:- $c : \left|z+\frac{1}{2}\right| = 1$

$$z+1 = 0$$

Here $z = -1$, $|-1+1/2| = |-1/2| = 1/2 < 1$, which lies inside c.

By cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\int_c \frac{e^z}{z-(-1)}\,dz = 2\pi i f(-1)$$

$$\int_C \frac{e^z}{z+1}\,dz = 2\pi i\,(1/e)$$

$$\int_C \frac{e^z}{z+1}\,dz = \frac{2\pi i}{e}$$

Here, $f(z) = e^z$

$f(-1) = e^{-1}$

$= 1/e$

(4) $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz, \quad c: |z| = 3$

Soln:-

$c: |z| = 3$

$(z-1)(z-2) = 0$

Here $z = 1$, $|1| = 1 < 3$, which lies inside c.

$z = 2$, $|2| = 2 < 3$ which lies inside c

By using partial fraction method,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$1 = A(z-2) + B(z-1) \rightarrow$

Put $z = 1$,

$1 = A(1-2) + B(1-1)$

$1 = -A \Rightarrow A = -1$

put $z = 2$

$\Rightarrow 1 = A(2-2) + B(2-1)$

$1 = A(0) + B(1)$

$1 = B$

$B = 1$

$f(z) = \sin\pi z^2 + \cos\pi z^2$

$f(z) = \sin\pi + \cos\pi$

$f(z) = 0 - 1 = -1$

$f(2) = \sin\pi(2)^2 + \cos\pi(2)^2$

$f(2) = 0 + 1 = 1$

$\sin n\pi = 0$

$\cos\pi, 3\pi, 5\pi = -1$

$\cos 2\pi, 4\pi, 6\pi = 1$

$\cos n\pi = (-1)^n$

Now,

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$$

$$\Rightarrow \int_C -\frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)}\,dz + \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)}\,dz$$

$\Rightarrow -2\pi i f(1) + 2\pi i f(2)$

$\Rightarrow -2\pi i(-1) + 2\pi i(1)$

$\Rightarrow 2\pi i + 2\pi i$

$\Rightarrow 4\pi i$

5) Evaluate $\int_C \frac{dz}{z^2-2z}$, where c is the circle $|z|=1$

Soln:- $\int_C \frac{dz}{z^2-2z} = \int_C \frac{dz}{z(z-2)}$

Given :- $c : |z| = 1$

$\therefore z(z-2) = 0$

Here $z=0$, $|0| = 0 < 1$, which lies inside c.

$z=2$, $|2| = 2 > 1$, which lies inside c.

$$\int_C \frac{dz}{z(z-2)} = \int_C \frac{\frac{1}{z-2}}{z}\,dz \rightarrow (1)$$

By cauchy's integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$(1) \Rightarrow \int \frac{dz}{z^2+2z} = 2\pi i f(0)$$

$$= 2\pi i(-1/2) = -\pi i$$

6) Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$ where c is the circle $|z-2| = 1/2$

Given :- $c : |z-2| = 1/2$

$(z-1)(z-2) = 0$

Here $z=1$, $|1-2| = |-1| = 1 > 1/2$ which lies outside c

$z=2$, $|2-2| = |0| = 0 < 1/2$, which lies inside c

$$\int_c \frac{z\,dz}{(z-1)(z-2)^2} = \int_c \frac{\frac{z}{z-1}}{(z-1)^2}\,dz$$

By cauchy's Integral formula,

$$\int_c \frac{f(z)}{(z-a)^2}\,dz = 2\pi i f'(a)$$

$$(1) \Rightarrow \int \frac{z\,dz}{(z-1)(z-2)^2} = 2\pi i f'(2)$$

$$= 2\pi i(-1)$$

$$= -2\pi i$$