



UNIT 4 Fourier Series and Fourier Transform

PROBLEMS UNDER FOURIER SERIES IN (0, 2π).

① find the Fourier series of $f(x) = \frac{\pi-x}{2}$;

$0 \leq x \leq 2\pi$. Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$.

Solution:

$f(x)$ is defined in $(0, 2\pi)$.

The Fourier series for $f(x)$ in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi-x}{2} \right) dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\pi-x) dx$$

$$= \frac{1}{2\pi} \left[\pi x - \frac{x^2}{2} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[2\pi^2 - \frac{(2\pi)^2}{2} - 0 - 0 \right]$$

$$= \frac{1}{2\pi} \left[2\pi^2 - \frac{4\pi^2}{2} \right] = \frac{1}{2\pi} (2\pi^2 - 2\pi^2) = \frac{1}{2\pi} (0)$$

$$\boxed{a_0 = 0}$$



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$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi-x}{2}\right) \cos nx \, dx \\
 &= \frac{1}{2\pi} \left[(\pi-x) \left(\frac{\sin nx}{n} \right) - (0-1) \left(-\frac{\cos nx}{n^2} \right) \right]_0^{2\pi} \\
 &= \frac{-1}{2\pi n^2} [\cos nx]_0^{2\pi} \\
 &= \frac{-1}{2\pi n^2} ([\cos 2\pi - \cos 0]) \\
 &= \frac{-1}{2\pi n^2} [1-1] = \frac{-1}{2\pi n^2} (0)
 \end{aligned}$$

$$a_n = 0$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi-x}{2}\right) \sin nx \, dx \\
 &= \frac{1}{2\pi} \int_0^{2\pi} (\pi-x) \sin nx \, dx
 \end{aligned}$$



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$$\begin{aligned}&= \frac{1}{2\pi} \left[(\pi - x) \left(-\frac{\cos nx}{n} \right) - (0-0) \left(\frac{\sin nx}{n^2} \right) \right]_0^{2\pi} \\&= \frac{-1}{2\pi n} [(\pi - x) \cos nx]_0^{2\pi} \\&= \frac{-1}{2\pi n} [(\pi - 2\pi) \cos 2\pi n - (\pi - 0) \cos 0] \\&= \frac{-1}{2\pi n} [-\pi(-1) - \pi(1)] = \frac{-1}{2\pi n} (-2\pi)\end{aligned}$$

$$b_n = \frac{1}{n}$$

Substitute the values of a_0 , a_n , b_n in ① we get

$$f(x) = \frac{a_0}{2} + 0 + \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \quad - \quad ②$$

$$\text{To deduce } 1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$$

Since $f(x)$ contains sine term, put $x = \frac{\pi}{2}$ in ② we get

$$f\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \quad - \quad ③$$

To find $f\left(\frac{\pi}{2}\right)$:

$$f(x) = \frac{\pi - x}{2}, \quad 0 < x < 2\pi$$



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Here $x = \frac{\pi}{2}$ lies inside $(0, 2\pi)$. Hence $f(x)$ continues at $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi - \pi/2}{2} = \frac{\pi/2}{2} = \frac{\pi}{4}$$

Substitute the value of $f\left(\frac{\pi}{2}\right)$ in ③, we get

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2}$$
$$\therefore \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} = \frac{\pi}{4}$$

$$\sin \frac{\pi}{2} + \frac{1}{2} \sin \frac{2\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} + \dots = \frac{\pi}{4}$$

$$1 + 0 + \frac{1}{3} (-1) + \dots = \frac{\pi}{4}$$

Ans:

$$\boxed{1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}}$$

2) Find the Fourier series for $f(x) = x(2\pi - x)$,
 $0 < x < 2\pi$

$$\text{i)} \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

$$\text{ii)} \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$$

$$\text{iii)} \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots = \frac{\pi^2}{8}$$



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Soln:

$f(x)$ is defined in $(0, 2\pi)$

The Fourier series for $f(x)$ in the interval $(0, 2\pi)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots \textcircled{1}$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (2\pi x - x^2) dx$$

$$= \frac{1}{\pi} \left[2\pi \frac{x^2}{2} - \frac{x^3}{3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\pi(2\pi)^2 - \frac{(2\pi)^3}{3} - 0 - 0 \right]$$

$$= \frac{1}{\pi} \left[4\pi^3 - \frac{8\pi^3}{3} \right]$$

$$= \frac{1}{\pi} \left[\frac{12\pi^3 - 8\pi^3}{3} \right]$$

$$\boxed{a_0 = \frac{4\pi^2}{3}}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (2\pi x - x^2) \cos nx dx$$



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$$\begin{aligned} &= \frac{1}{\pi} \left[(2\pi - 2x) \left(\frac{\sin nx}{n} \right) - (2\pi - 2x) \left(\frac{-\cos nx}{n^2} \right) + (0-2x) \left(\frac{\cos nx}{n^2} \right) \right]_0^{2\pi} \\ &= \frac{1}{\pi} \left[(2\pi - 2x) \left(\frac{\cos nx}{n^2} \right) \right]_0^{2\pi} \\ &= \frac{1}{\pi} \left[(2\pi - 4\pi) \left(\frac{\cos 2\pi}{n^2} \right) - 2\pi \left(\frac{\cos 0}{n^2} \right) \right] \\ &= \frac{1}{\pi} \left[-2\pi \frac{1}{n^2} - 2\pi \frac{1}{n^2} \right] \\ &= \frac{1}{\pi} \left(-\frac{4\pi}{n^2} \right) \end{aligned}$$

$$a_n = \boxed{\frac{-4}{n^2}}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \\ &= \frac{1}{\pi} \int_0^{2\pi} (2\pi x - x^2) \sin nx dx \\ &= \frac{1}{\pi} \left[(2\pi x - x^2) \left(\frac{-\cos nx}{n} \right) - (2\pi - 2x) \left(\frac{-\sin nx}{n^2} \right) \right]_0^{2\pi} \\ &= \frac{1}{\pi} \left[(2\pi \cdot 2 - 2^2) \left(\frac{-\cos 2\pi}{n} \right) - (2\pi - 2 \cdot 2) \left(\frac{-\sin 2\pi}{n^2} \right) \right] \end{aligned}$$

$$= \frac{1}{\pi} \left[(2\pi \cdot 2 - 2^2) \left(\frac{-\cos 2\pi}{n} \right) - 2 \left(\frac{\cos 0}{n^2} \right) \right]^{2\pi}$$



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$$= \frac{1}{\pi} \left[\frac{-2}{n^3} + \frac{2}{n^3} \right]$$

$$bn = 0$$

Substitute the values of a_0, a_n, b_n in ①, we get

$$f(x) = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx \quad - ②$$

i) To deduce $\frac{1}{1^2} + \frac{1}{2^2} + \dots = \frac{\pi^2}{6}$

Put $x=0$ in ②, we get

$$\begin{aligned} f(0) &= \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 0 \\ &= \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned} \quad - ③$$

To find $f(0)$:

We have $f(x) = 2\pi(x - x^2)$, $0 < x < 2\pi$

$x=0$ is end of $(0, 2\pi)$

$$f(0) = 0$$

$$\begin{aligned} f(2\pi) &= 2\pi(2\pi) - (2\pi)^2 \\ &= 4\pi^2 - 4\pi^2 \\ &= 0 \end{aligned}$$

$$f(0) = f(2\pi) = 0 \quad \text{Ms.C.Saranya, AP/Maths}$$



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Substitute the value of $f(0)$ in ③ we get,

$$0 = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6} \quad \textcircled{a}$$

i) To deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

Put $x=\pi$ in ② we get

$$f(\pi) = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi$$

$$f(\pi) = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \quad \textcircled{4}$$

To find $f(\pi)$:

$x=\pi$ lies inside $(0, 2\pi)$

$\therefore f(x)$ is continuous at $x=\pi$

$$f(\pi) = 2\pi^2 - \pi^2$$

$$f(\pi) = \pi^2$$

Substitute the value of $f(\pi)$ in ④, we get



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$$\pi^2 = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$
$$- 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \pi^2 - \frac{2\pi^2}{3}$$
$$= \frac{2\pi^2 - 2\pi^2}{3}$$
$$= \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots = \frac{-\pi^2}{12}$$

$$\underline{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}} \quad \textcircled{b}$$

iii) To deduce $\frac{1}{1^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{8}$

Adding \textcircled{a} and \textcircled{b} , we get

Ans:
$$\boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}}$$



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Homework sums :

① Find the Fourier series for $f(x) = x$ in $0 < x < 2\pi$. Hence deduce $1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$

Ans :
$$f(x) = \pi - 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

② Find the Fourier series of $f(x) = (\pi - x)^2$ in $(0, 2\pi)$ of periodicity 2π

Ans :
$$f(x) = \frac{\pi^2}{3} + 4 \left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots \right]$$

③ Find the Fourier series for $f(x) = x^2$, $0 < x < 2\pi$. Hence deduce

a) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

b) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

c) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Ans :
$$f(x) = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$