UNIT 4 Fourier Series and Fourier Transform

PROBLEMS UNDER FOURIER SERIES IN $(0, 2\pi)$.

(1) Find the Fourier series of $f(x) = \frac{\pi - x}{2}$; $0 < x < 2\pi$. Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$.

Solution:

$f(x)$ is defined in $(0, 2\pi)$.

The fourier series for $f(x)$ in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -(1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right) dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\,dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi^2 - \frac{(2\pi)^2}{2} - 0 - 0\right]$$

$$= \frac{1}{2\pi}\left[2\pi^2 - \frac{4\pi^2}{2}\right] = \frac{1}{2\pi}(2\pi^2 - 2\pi^2) = \frac{1}{2\pi}(0)$$

$$\boxed{a_0 = 0}$$

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$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)\cos nx\, dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (0-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n^2}\left[\cos nx\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n^2}\left(\left[\cos 2n\pi - \cos 0\right]\right)$$

$$= \frac{-1}{2\pi n^2}\left[1-1\right] = \frac{-1}{2\pi n^2}(0)$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)\sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx\, dx$$

$$= \frac{1}{2\pi}\left[(\pi-x)\left(-\frac{\cos nx}{n}\right)-(0-1)\left(\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{-1}{2n\pi}\left[(\pi-x)\cos nx\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n}\left[(\pi-2\pi)\cos 2n\pi-(\pi-0)\cos 0\right]$$

$$= \frac{-1}{2\pi n}\left[-\pi(1)-\pi(1)\right] \qquad = \frac{-1}{2n\pi}(-2\pi)$$

$$\boxed{b_n = \frac{1}{n}}$$

Substitute the values of a_0, a_n, b_n in ① we get

$$f(x) = \frac{0}{2}+0+\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n}\sin nx \quad — ②$$

To deduce $1-\frac{1}{3}+\frac{1}{5}-\dots = \frac{\pi}{4}$

Since $f(x)$ contains sine term, put $x = \frac{\pi}{2}$ in ② we get

$$f\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2} \quad — ③$$

To find $f\left(\frac{\pi}{2}\right)$:

$$f(x) = \frac{\pi-x}{2}, \quad 0 < x < 2\pi$$

Here $x = \frac{\pi}{2}$ lies inside $(0, 2\pi)$. Hence $f(x)$ continues at $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi - \pi/2}{2} = \frac{\pi/2}{2} = \frac{\pi}{4}$$

Substitute the value of $f\left(\frac{\pi}{2}\right)$ in ③, we get

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} = \frac{\pi}{4}$$

$$\sin \frac{\pi}{2} + \frac{1}{2} \sin \frac{2\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} + \dots = \frac{\pi}{4}$$

$$1 + 0 + \frac{1}{3}(-1) + \dots = \frac{\pi}{4}$$

Ans: $\boxed{1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}.}$

2) Find the Fourier series for $f(x) = x(2\pi - x)$; $0 \le x < 2\pi$

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots = \frac{\pi^2}{8}$

Soln: $f(x)$ is defined in $(0, 2\pi)$

The Fourier series for $f(x)$ in the interval $(0, 2\pi)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\,dx$$

$$= \frac{1}{\pi}\left[2\pi \frac{x^2}{2} - \frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\pi(2\pi)^2 - \frac{(2\pi)^3}{3} - 0 - 0\right]$$

$$= \frac{1}{\pi}\left[4\pi^3 - \frac{8\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{12\pi^3 - 8\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{4\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\cancel{(2\pi x - x^2)\left(\frac{\sin nx}{n}\right)} - (2\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + \cancel{(0-2)}\left(\right.\right.$$

$$= \frac{1}{\pi}\left[(2\pi - 2x)\left(\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(2\pi - 4\pi)\left(\frac{\cos 2n\pi}{n^2}\right) - 2\pi\frac{\cos 0}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-2\pi\frac{1}{n^2} - 2\pi\frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left(-\frac{4\pi}{n^2}\right)$$

$$\boxed{a_n = \frac{-4}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2n\pi - x^2)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - \cancel{(2\pi - 2x)\left(\frac{-\sin nx}{n^2}\right)} + (-\right.$$

$$= \frac{1}{\pi}\left[(2n\pi - x^2)\left(\frac{-\cos nx}{n}\right) - 2\left(\frac{\cos nx}{n^3}\right)\right]^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-2}{n^3} + \frac{2}{n^3}\right]$$

$$\boxed{b_n = 0}$$

Substitute the values of a_0, a_n, b_n in (1), we get

$$f(x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx \quad — (2)$$

i) To deduce $\frac{1}{1^2} + \frac{1}{2^2} + \dots = \frac{\pi^2}{6}$

Put $x = 0$ in (2), we get

$$f(0) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{1}{n^2}\cos 0$$

$$= \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{1}{n^2} \quad — (3)$$

To find $f(0)$:

We have $f(x) = 2\pi x - x^2$; $0 < x < 2\pi$

$x = 0$ is end of $(0, 2\pi)$

$$f(0) = 0$$

$$f(2\pi) = 2\pi(2\pi) - (2\pi)^2$$

$$= 4\pi^2 - 4\pi^2$$

$$= 0$$

$$f(0) = f(2\pi) = 0$$

Substitute the value of $f(0)$ in ③ we get,

$$0 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \quad \text{— (a)}$$

ii) To deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

Put $x = \pi$ in ② we get

$$f(\pi) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi$$

$$f(\pi) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n \quad \text{— (4)}$$

To find $f(\pi)$:

$x = \pi$ lies inside $(0, 2\pi)$

$\therefore f(x)$ is continuous at $x = \pi$

$$f(\pi) = 2\pi^2 - \pi^2$$

$$f(\pi) = \pi^2$$

Substitute the value of $f(\pi)$ in ④, we get

$$\pi^2 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^2}{n^2}$$

$$-4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \pi^2 - \frac{2\pi^2}{3}$$

$$= \frac{3\pi^2 - 2\pi^2}{3}$$

$$= \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots = \frac{-\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12} \quad \text{--- (b)}$$

iii) To deduce $\frac{1}{1^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{8}$

Adding (a) and (b), we get

Ans: $\boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}}$

Homework sums:

(1) Find the Fourier series for $f(x) = x$ in $0 < x < 2\pi$. Hence deduce $1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$

Ans: $\boxed{f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{1}{n} \sin nx}$

(2) Find the Fourier series of $f(x) = (\pi - x)^2$ in $(0, 2\pi)$ of periodicity 2π

Ans: $\boxed{f(x) = \frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots\right]}$

(3) Find the Fourier series for $f(x) = x^2$; $0 < x < 2\pi$. Hence deduce

a) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

b) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

c) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Ans: $f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$