



UNIT-V LAPLACE TRANSFORM

PROPERTIES:

Change of Scale property:

✓ If $\mathcal{L}\{f(t)\} = F(s)$, then
 $\mathcal{L}[f(at)] = \frac{1}{a} F(s/a)$.

Proof:

We know that,

$$\mathcal{L}[f(at)] = \int_0^{\infty} e^{-st} f(at) dt$$

$$\text{put } at = x \Rightarrow t = x/a$$

$$\mathcal{L}[f(at)] = \int_0^{\infty} e^{-s(x/a)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^{\infty} e^{-s(x/a)} f(x) dx$$

$$= \frac{1}{a} \int_0^{\infty} e^{-(s/a)x} f(x) dx$$

$$= \frac{1}{a} \int_0^{\infty} e^{-(s/a)t} f(t) dt$$

$$= \frac{1}{a} F(s/a)$$



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First Shifting property:

If $\mathcal{L}\{f(t)\} = F(s)$ then -

$$(i) \mathcal{L}[e^{-at}f(t)] = \left\{ \mathcal{L}[f(t)] \right\}_{s \rightarrow s+a} = F(s+a)$$

$$(ii) \mathcal{L}[e^{at}f(t)] = \left\{ \mathcal{L}[f(t)] \right\}_{s \rightarrow s-a} = F(s-a)$$

Proof:

(i) We know that,

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

$$\begin{aligned} \mathcal{L}[e^{-at}f(t)] &= \int_0^{\infty} e^{-st} [e^{-at}f(t)] dt \\ &= \int_0^{\infty} e^{-(s+a)t} f(t) dt \\ &= F(s+a) \end{aligned}$$

$$\begin{aligned} (ii) \mathcal{L}[e^{at}f(t)] &= \int_0^{\infty} e^{-st} [e^{at}f(t)] dt \\ &= \int_0^{\infty} e^{-(s-a)t} f(t) dt \\ &= F(s-a) \end{aligned}$$

Second Shifting property:

If $\mathcal{L}\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t \geq a \\ 0, & t < a \end{cases}$

then $\mathcal{L}[g(t)] = e^{-as} F(s)$.

Proof:

$$\begin{aligned} \mathcal{L}[g(t)] &= \int_0^{\infty} e^{-st} g(t) dt \\ &= \int_0^a e^{-st} g(t) dt + \int_a^{\infty} e^{-st} g(t) dt \end{aligned}$$



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$$\mathcal{L}[g(t)] = 0 + \int_a^{\infty} e^{-st} f(t-a) dt$$

$$= \int_a^{\infty} e^{-st} f(t-a) dt$$

$$\text{Put } t-a = u \Rightarrow dt = du$$

$$\text{When } t = a \Rightarrow u = 0$$

$$t \rightarrow \infty \Rightarrow u \rightarrow \infty$$

$$\mathcal{L}[g(t)] = \int_0^{\infty} e^{-s(u+a)} f(u) du$$

$$= \int_0^{\infty} e^{-us} e^{-as} f(u) du$$

$$= e^{-as} \int_0^{\infty} e^{-us} f(u) du$$

$$= e^{-as} \int_0^{\infty} e^{-st} f(t) dt \quad \text{Replace } u \rightarrow t$$

$$\mathcal{L}[g(t)] = e^{-as} F(s)$$

Laplace transforms of derivatives:

$$\text{If } \mathcal{L}[f(t)] = F(s) \text{ then}$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0).$$

Proof:

$$\mathcal{L}[f'(t)] = \int_0^{\infty} e^{-st} f'(t) dt$$

Integrating by parts we get,

$$= \left[e^{-st} f(t) \right]_0^{\infty} - \int_0^{\infty} f(t) (-s e^{-st}) dt$$



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$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \left[e^{-st} f(t) + s \int_0^{\infty} e^{-st} f(t) dt \right]_0^{\infty} \\ &= -f(0) + s \mathcal{L}\{f(t)\} \\ &= sF(s) - f(0) \end{aligned}$$

Corollary:

$$\text{Let } f''(t) = S^2 F(s) - S f(0) - f'(0)$$

$$\text{Let } \mathcal{L}[g'(t)] = S G(s) - g(0)$$

We know that,

$$\mathcal{L}[f'(t)] = S \mathcal{L}[f(t)] - f(0)$$

Replace $f(t) \rightarrow f'(t)$ & $f'(t) \rightarrow f''(t)$ & $f(0) \rightarrow f'(0)$

$$\begin{aligned} \Rightarrow \mathcal{L}[f''(t)] &= S \mathcal{L}[f'(t)] - f'(0) \\ &= S [S \mathcal{L}[f(t)] - f(0)] - f'(0) \\ &= S^2 \mathcal{L}[f(t)] - S f(0) - f'(0) \\ &= S^2 F(s) - S f(0) - f'(0) \end{aligned}$$

Laplace Transform of integrals:

$$\text{If } \mathcal{L}[f(t)] = F(s) \text{ then } \mathcal{L}\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$$

Proof:

$$\text{Let } g(t) = \int_0^t f(t) dt \text{ and } g(0) = 0$$

$$\text{Then } g'(t) = f(t)$$

$$\text{WKT } \mathcal{L}[g'(t)] = S \mathcal{L}[g(t)] - g(0)$$

$$= S \mathcal{L}[g(t)]$$

$$\Rightarrow \mathcal{L}[g(t)] = \frac{1}{s} \mathcal{L}[g'(t)]$$



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$$\Rightarrow \mathcal{L} \left[\int_0^t f(t) dt \right] = \frac{1}{s} \mathcal{L} [f(t)] \quad \left\{ \begin{array}{l} \because g(t) = \int_0^t f(t) dt \\ g'(t) = f(t) \end{array} \right.$$
$$\Rightarrow \mathcal{L} \left[\int_0^t f(t) dt \right] = \frac{F(s)}{s}$$

Derivative of Laplace Transform (or) Laplace transform of $t f(t)$:

If $\mathcal{L} [f(t)] = F(s)$ then

$$\mathcal{L} [t f(t)] = -\frac{d}{ds} F(s)$$

Proof: We know that,

$$\mathcal{L} [f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$
$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt$$
$$= \int_0^{\infty} \frac{\partial}{\partial s} (e^{-st}) f(t) dt$$
$$= \int_0^{\infty} -t e^{-st} f(t) dt$$
$$= - \int_0^{\infty} e^{-st} t f(t) dt$$
$$= - \mathcal{L} [t f(t)]$$
$$\Rightarrow \mathcal{L} [t f(t)] = -\frac{d}{ds} [F(s)]$$

In general,

$$\mathcal{L} [t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$$



UNIT-V LAPLACE TRANSFORM

Problems: $1+2-2 = [(1)7] 1$ - mixed (2)

Change of Scale property:

① Find $L[\sinh 3t]$ by using change of scale property

Soln:

$$L[\sinh ht] = \frac{1}{s^2 - h^2} = F(s)$$

$$L[\sinh 3t] = \frac{1}{3^2} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3} \cdot \frac{1}{\frac{s^2 - 9}{9}}$$

$$= \frac{1}{3} \cdot \frac{9}{s^2 - 9} = \frac{3}{s^2 - 9}$$

② Find $L(\cos 5t)$ using change of scale property?

Soln:

$$L(\cos t) = \frac{s}{s^2 + 1} = F(s)$$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5} \left[\frac{s/5}{\left(\frac{s}{5}\right)^2 + 1} \right]$$

$$= \frac{1}{5} \left[\frac{5s}{s^2 + 25} \right]$$

$$= \frac{s}{s^2 + 25}$$



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UNIT-V LAPLACE TRANSFORM

Problems: $1+2-2 = [(1)+] \downarrow$ inverted (8)

Change of Scale property:

① Find $L[\sinh 3t]$ by using change of scale property

Soln:

$$L[\sinh ht] = \frac{1}{s^2 - h^2} = F(s)$$

$$L[\sinh 3t] = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3} \frac{1}{\frac{s^2}{9} - 1}$$

$$= \frac{1}{3} \frac{1}{\frac{s^2 - 9}{9}}$$

$$= \frac{1}{3} \frac{9}{s^2 - 9} = \frac{3}{s^2 - 9}$$

② Find $L(\cos 5t)$ using change of scale property?

Soln:

$$L(\cos t) = \frac{s}{s^2 + 1} = F(s)$$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5} \left[\frac{s/5}{\left(\frac{s}{5}\right)^2 + 1} \right]$$

$$= \frac{1}{5} \left[\frac{5s}{s^2 + 25} \right]$$

$$= \frac{s}{s^2 + 25}$$



UNIT-V LAPLACE TRANSFORM

First Shifting theorem:

① Find $\mathcal{L}[e^{-3t} \sin^2 t]$

Proof:

$$\mathcal{L}[e^{-at} f(t)] = F(s+a)$$

$$\mathcal{L}[e^{-3t} \sin^2 t] = \mathcal{L}[\sin^2 t]_{s \rightarrow s+3}$$

$$= \mathcal{L}\left[\frac{1 - \cos 2t}{2}\right]_{s \rightarrow s+3}$$

$$= \frac{1}{2} \{ \mathcal{L}(1) - \mathcal{L}(\cos 2t) \}_{s \rightarrow s+3}$$

$$= \frac{1}{2} \left\{ \frac{1}{s} - \frac{s}{s^2 + 4} \right\}_{s \rightarrow s+3}$$

$$= \frac{1}{2} \left\{ \frac{1}{s+3} - \frac{s+3}{(s+3)^2 + 4} \right\}$$

$$= \frac{1}{2} \left\{ \frac{4}{(s+3)[(s+3)^2 + 4]} \right\}$$

$$= \frac{2}{(s+3)[(s+3)^2 + 4]}$$

② Find $\mathcal{L}(t^2 e^{-2t})$.

Soln:

$$\mathcal{L}[e^{-at} f(t)] = F(s+a)$$

$$\mathcal{L}[e^{-2t} t^2] = \mathcal{L}(t^2)_{s \rightarrow s+2}$$

$$= \left[\frac{2}{s^3} \right]_{s \rightarrow s+2}$$

$$= \frac{2}{(s+2)^3}$$



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③ Find $L[e^{2t} \cos 5t]$.

Soln:

$$L[e^{2t} \cos 5t] = L[\cos 5t]_{s \rightarrow s-2}$$

$$= \left[\frac{s}{s^2 + 25} \right]_{s \rightarrow s-2}$$

$$= \frac{s-2}{(s-2)^2 + 25}$$

Second Shifting theorem:

① Find $L[f(t)]$ where $f(t) = \begin{cases} 0, & 0 \leq t < 2 \\ 3, & t \geq 2 \end{cases}$

Soln:

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^2 e^{-st} f(t) dt + \int_2^{\infty} e^{-st} f(t) dt$$

$$= 0 + \int_2^{\infty} e^{-st} \cdot 3 dt$$

$$= 3 \left[\frac{e^{-st}}{-s} \right]_2^{\infty}$$

$$= -\frac{3}{s} [e^{-\infty} - e^{-2s}]$$

$$= \frac{3e^{-2s}}{s}$$

② Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$$

Soln:

$$L\{f(t)\} = \int_0^{\pi} e^{-st} \sin t dt + \int_{\pi}^{\infty} e^{-st} \cdot 0 dt$$



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$$\begin{aligned}
 \mathcal{L}[f(t)] &= \int_0^{\infty} e^{-st} f(t) dt \\
 &= \int_0^{\pi} e^{-st} f(t) dt + \int_{\pi}^{\infty} e^{-st} 0 dt \\
 &= \left[\frac{e^{-st}}{s^2+1} (-s \sin t - \cos t) \right]_0^{\pi} \\
 &= \frac{e^{-s\pi}}{s^2+1} (-s \sin \pi - \cos \pi) + \frac{1}{s^2+1} \\
 &= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}
 \end{aligned}$$

Laplace Transforms of Derivatives:

(1) Find $\mathcal{L}[t \sin at]$

Soln:

$$\mathcal{L}[t \sin at] =$$

$$f(t) = t \sin at$$

$$f'(t) = at \cos at + \sin at$$

$$f''(t) = a[-at \sin at + \cos at] + a \cos at$$

$$f''(t) = 2a \cos at - a^2 t \sin at$$

$$f(0) = 0, f'(0) = 0$$

$$\mathcal{L}[f''(t)] = s^2 \mathcal{L}[f(t)] - s f(0) - f'(0)$$

$$\mathcal{L}[2a \cos at - a^2 t \sin at] = s^2 \mathcal{L}[t \sin at] - s(0) - 0$$

$$\Rightarrow 2a \mathcal{L}[\cos at] - a^2 \mathcal{L}[t \sin at] = s^2 \mathcal{L}[t \sin at]$$

$$\Rightarrow (s^2 + a^2) \mathcal{L}[t \sin at] = 2a \mathcal{L}[\cos at]$$

$$\Rightarrow (s^2 + a^2) \mathcal{L}[t \sin at] = 2a \cdot \frac{s}{a^2 + s^2}$$

$$\mathcal{L}[t \sin at] = \frac{2as}{(s^2 + a^2)^2}$$



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④ Find $\mathcal{L} \left[\frac{\sin 3t}{t} \right]$

Soln:

$$\mathcal{L} \left[\frac{f(t)}{t} \right] = \int_s^\infty F(s) ds = \int_s^\infty \mathcal{L}[f(t)] ds$$

$$\mathcal{L} \left[\frac{\sin 3t}{t} \right] = \int_s^\infty \mathcal{L}(\sin 3t) ds$$

$$= \int_s^\infty \left(\frac{3}{s^2 + 9} \right) ds$$

$$= \int_s^\infty \frac{3}{s^2 + 3^2} ds$$

$$= 3 \cdot \frac{1}{3} \left[\tan^{-1} \left(\frac{s}{3} \right) \right]_s^\infty \quad \left(\because \int \frac{dx}{x^2 + a^2} \right)$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3) \quad = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

$$= \pi/2 - \tan^{-1}(s/3)$$

$$= \cot^{-1}(s/3)$$