CONVOLUTION :

Defn.

If $f(t)$ and $g(t)$ are two functions defined for $t \geq 0$ then the convolution of $f(t)$ & $g(t)$ is defined as $f(t) * g(t) = (f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$

NOTE : $f(t) * g(t) = g(t) * f(t)$

CONVOLUTION THEOREM :

If $f(t)$ & $g(t)$ are two laplace transformable functions defined for $t \geq 0$ then

$L[f(t) * g(t)]$ is given by

$L[f(t) * g(t)] = L[f(t)] * L[g(t)]$

$L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

① Using Convolution theorem find $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$

Soln:

$L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\dfrac{s}{s^2+a^2} \cdot \dfrac{1}{s^2+a^2}\right]$

$= L^{-1}\left[\dfrac{s}{s^2+a^2}\right] \cdot L^{-1}\left[\dfrac{1}{s^2+a^2}\right]$ [By Convoln. thm

$= \cos at \cdot \dfrac{\sin at}{a}$

$= \dfrac{1}{a} \cos at \cdot \sin at$

$= \dfrac{1}{a}\displaystyle\int_0^t \cos au \, \sin a(t-u) \, du$ [By Convoln. defn]

$= \dfrac{1}{2a}\displaystyle\int_0^t [\sin at - \sin(2au-at)] \, du$

$= \dfrac{1}{2a}\left\{ u\sin at + \dfrac{\cos(2au-at)}{2a} \right\}_0^t$

$= \dfrac{1}{2a}\left[t\sin at + \dfrac{\cos at}{2a} - 0 - \dfrac{\cos at}{2a} \right]$

$= \dfrac{1}{2a} t\sin at$

(side notes)

$\sin A \sin B = \dfrac{1}{2}[\sin(A+B) - \sin(A-B)]$

Find the inverse LT by convolution thm. $\dfrac{1}{s^2(s+5)}$

$L^{-1}\left[\dfrac{1}{s^2(s+5)}\right] = L^{-1}\left[\dfrac{1}{s^2}\right] * L^{-1}\left[\dfrac{1}{s+5}\right]$

$= t * e^{-5t}$

$= \displaystyle\int_0^t u\, e^{-5(t-u)}\, du$

$= e^{-5t} \displaystyle\int_0^t u\, e^{5u}\, du$

$= e^{-5t}\left[\dfrac{u e^{5u}}{5} - \dfrac{e^{5u}}{25}\right]_0^t$

$= e^{-5t}\left[\dfrac{t e^{5t}}{5} - \dfrac{e^{5t}}{25} - \left(-\dfrac{1}{25}\right)\right]$

$= \dfrac{t}{5} - \dfrac{1}{25} + \dfrac{e^{-5t}}{25}$

$= \dfrac{1}{25}\left[5t + e^{-5t} - 1\right]$

③ using convolution theorem find

i) $\dfrac{1}{(s^2+a^2)^2}$ soln: $\dfrac{1}{2a^2}\left[\dfrac{\sin at}{a} - t\cos at\right]$

ii) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ soln: $\dfrac{a\sin at - b\sin bt}{a^2-b^2}$