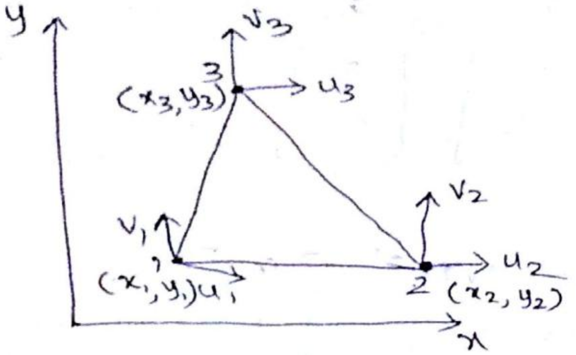
Shape Function Derivation for the CST element :-

(V)



Consider a typical CST element with nodes 1, 2, 3, Nodal displacements be u_1, u_2, u_3, v_1, v_2 and v_3.

$$\text{Displacement } \{u\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

u_1, u_2, u_3 → displacement along x axis

v_1, v_2, v_3 → displacement along y axis.

CST element has got two degrees of freedom at each node (u, v), Total dof = 6, Hence it is six generalized coordinates

Let $u = a_1 + a_2 x + a_3 y$

$v = a_4 + a_5 x + a_6 y$.

now apply the nodal conditions,

$u_1 = a_1 + a_2 x_1 + a_3 y_1$

$u_2 = a_1 + a_2 x_2 + a_3 y_2$

$u_3 = a_1 + a_2 x_3 + a_3 y_3$

$v_1 = a_4 + a_5 x_1 + a_6 y_1$

$v_2 = a_4 + a_5 x_2 + a_6 y_2$

$v_3 = a_4 + a_5 x_3 + a_6 y_3$

In matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Let $D = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$

$D^{-1} = \dfrac{C^T}{|D|}$. $C \rightarrow$ co-factors of Matrix D

$$C = \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{vmatrix}$$

$$|D| = (x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2)$$

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$2A = (x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2).$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_2 - x_3) & (x_2 - x_1) \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A}\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

WKT,

$$u = a_1 + a_2 x + a_3 y$$

$$u = [1 \quad x \quad y]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$= [1 \quad x \quad y]\frac{1}{2A}\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A}\left[p_1 + q_1 x + r_1 y, \quad p_2 + q_2 x + r_2 y, \quad p_3 + q_3 x + r_3 y \right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \longrightarrow (1)$$

similary

$$v = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \longrightarrow (2)$$

$$N_1 = \frac{p_1 + q_1 x + r_1 y}{2A}$$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

rrange the equation (1) & (2)

$$u = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

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